Principles of KLM-style Defeasible Description Logics

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The past 25 years have seen many attempts to introduce defeasible-reasoning capabilities into a description logic setting. Many, if not most, of these attempts are based on preferential extensions of description logics, with a significant number of these, in turn, following the so-called KLM approach to defeasible reasoning initially advocated for propositional logic by Kraus, Lehmann, and Magidor. Each of these attempts has its own aim of investigating particular constructions and variants of the (KLM-style) preferential approach. Here our aim is to provide a comprehensive study of the formal foundations of preferential defeasible reasoning for description logics in the KLM tradition.

We start by investigating a notion of *defeasible subsumption* in the spirit of defeasible conditionals as studied by Kraus, Lehmann, and Magidor in the propositional case. In particular, we consider a natural and intuitive semantics for defeasible subsumption, and we investigate KLM-style syntactic properties for both *preferential* and *rational* subsumption. Our contribution includes two representation results linking our semantic constructions to the set of preferential and rational properties considered. Besides showing that our semantics is appropriate, these results pave the way for more effective decision procedures for defeasible reasoning in description logics. Indeed, we also analyse the problem of non-monotonic reasoning in description logics at the level of *entailment* and present an algorithm for the computation of *rational closure* of a defeasible knowledge base. Importantly, our algorithm relies completely on classical entailment and shows that the computational complexity of reasoning over defeasible knowledge bases is no worse than that of reasoning in the underlying classical DL $\mathcal{ALC}$.

CCS Concepts: • **Theory of computation → Logic**; **Description logics**; **Automated reasoning**;

Additional Key Words and Phrases: Non-monotonic reasoning, defeasible subsumption, preferential semantics, rational closure

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1 INTRODUCTION

Description logics (DLs) [1] are central to many modern AI and database applications, since they provide the logical foundation of formal ontologies. Yet, as classical formalisms, DLs do not allow for the proper representation of and reasoning with defeasible information, as shown up in the following example, adapted from Giordano et al. [54]: Students do not get tax invoices; employed students do; employed students who are also parents do not. From a naïve (classical) formalisation of this scenario, one concludes that the notion of employed student is an oxymoron, and, consequently, the concept of employed student is unsatisfiable. But while concept unsatisfiability has been investigated extensively in ontology debugging and repair [72, 83], our research problem here goes beyond that, as will become clear in the upcoming sections.

Endowing DLs with defeasible reasoning features is therefore a promising endeavour from the point of view of applications of knowledge representation and reasoning. Indeed, the past 25 years have witnessed many attempts to introduce defeasible-reasoning capabilities in a DL setting, drawing on a well-established body of research on non-monotonic reasoning (NMR). These comprise the so-called preferential approaches [27, 28, 30, 45, 48, 54, 55, 59, 60, 77, 78], circumscription-based ones [9, 10, 84], amongst others [2, 3, 8, 51, 64–66, 73, 76, 86].

Preferential extensions of DLs turn out to be particularly promising, mainly because they are based on an elegant, comprehensive, and well-studied framework for non-monotonic reasoning in the propositional case proposed by Kraus, Lehmann, and Magidor [67, 70] and often referred to as the *KLM approach*. Such a framework is valuable for a number of reasons. First, it provides for a thorough analysis of some formal properties that any consequence relation deemed as appropriate in a non-monotonic setting ought to satisfy. Such formal properties, which resemble those of a Gentzen-style proof system (see Section 3.1), play a central role in assessing how intuitive the obtained results are and enable a more comprehensive characterisation of the introduced non-monotonic conditional from a logical point of view. Second, the KLM approach allows for many decision problems to be reduced to classical entailment checking, sometimes without blowing up the computational complexity compared to the underlying classical case. Finally, it has a well-known connection with the AGM-approach to belief revision [53, 81] and with frameworks for reasoning under uncertainty [7, 52]. It is therefore reasonable to expect that most, if not all, of the aforementioned features of the KLM approach should transfer to KLM-based extensions of DLs, too.

Following the motivation laid out above, several extensions to the KLM approach to description logics have been proposed recently [27, 30, 33, 35, 38, 39, 45, 48, 54, 55, 59, 60, 74, 87], each of them investigating particular constructions and variants of the preferential approach. However, here our aim is to provide a *comprehensive* study of the formal foundations of preferential defeasible reasoning in DLs. By that, we mean (*i*) defining a general and intuitive *semantics*; (*ii*) showing that the corresponding *representation results* (in the KLM sense of the term) hold, linking our semantic constructions with the KLM-style set of properties; and (*iii*) presenting an appropriate analysis of *entailment* in the context of ontologies with defeasible information with an associated decision procedure that is implementable.

In the remainder of the article, we shall take the following route: After providing the required background on the DL, we consider in this work as well as fixing the notation (Section 2), we introduce the notion of defeasible subsumption along with a set of KLM-inspired properties it ought to satisfy (Section 3). In particular, using an intuitive semantics for the idea that "usually, an element of the class C is also an element of the class D", we provide a characterisation (via representation results) of two important classes of defeasible statements, namely, preferential and rational subsumption. In Section 4, we start by investigating two obvious candidates for the notion of entailment in the context of defeasible DLs, namely, preferential and modular entailment. These turn out *not* to have all properties seen as important in a non-monotonic DL setting, mimicking a similar result in the propositional case [70]. Therefore, we propose a notion of rational entailment and show that it is the definition of consequence we are looking for. We take this definition further by exploring the relationship that rational entailment has with both Lehmann and Magidor's [70] definition of rational closure and the more recent algorithm by Casini and Straccia [45] for its computation (Section 5). After a discussion of, and comparison with, related work (Section 6), we conclude with a summary of our contributions and some directions for further exploration. Proofs of our results can be found in the Appendix. Most of the results we present here have already been published in two technical reports [23, 24], and already cited and used in other publications (see Section 6). This article offers an expanded and more comprehensive presentation.

2 LOGICAL PRELIMINARIES

DLs [1] are decidable fragments of first-order logic with interesting properties and a variety of applications. There is a whole family of description logics, an example of which is $\mathcal{ALC}$ and on which we shall focus in the present article.[1]

The (concept) language of $\mathcal{ALC}$ is built upon a finite set of atomic *concept names* C, a finite set of *role names* R (a.k.a. *attributes*) and a finite set of *individual names* I such that C, R and I are pairwise disjoint. In our scenario example, we can have for instance $\mathsf{C} = \{\mathsf{Employee}, \mathsf{Company}, \mathsf{Student}, \mathsf{EmpStud}, \mathsf{Parent}, \mathsf{Tax}\}$, $\mathsf{R} = \{\mathsf{pays}, \mathsf{empBy}, \mathsf{worksFor}\}$, and $\mathsf{I} = \{\mathsf{john}, \mathsf{ibm}, \mathsf{mary}\}$, with the respective obvious intuitions. With $A, B, \ldots$, we denote atomic concepts, with $r, s, \ldots$ role names, and with $a, b, \ldots$ individual names. Complex concepts are denoted with $C, D, \ldots$ and are built using the constructors $\neg$ (complement), $\sqcap$ (concept conjunction), $\sqcup$ (concept disjunction), $\forall$ (value restriction) and $\exists$ (existential restriction) according to the following grammar rules:

$$C ::= \top \mid \bot \mid \mathsf{C} \mid (\neg C) \mid (C \sqcap C) \mid (C \sqcup C) \mid (\exists r.C) \mid (\forall r.C)$$

With $\mathcal{L}$, we denote the *language* of all $\mathcal{ALC}$ concepts, which is understood as the smallest set of symbol sequences generated according to the rules above. When writing down concepts of $\mathcal{L}$, we follow the usual convention and omit parentheses whenever they are not essential for disambiguation. Examples of $\mathcal{ALC}$ concepts in our scenario are $\mathsf{Student} \sqcap \mathsf{Employee}$ and $\neg\exists\mathsf{pays}.\mathsf{Tax}$.

The semantics of $\mathcal{ALC}$ is the standard set-theoretic Tarskian semantics. An *interpretation* is a structure $\mathcal{I} =_{\text{def}} \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping concept names A to subsets $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, role names r to binary relations $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$, and individual names a to elements of the domain $\Delta^{\mathcal{I}}$, i.e., $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

Figure 1 depicts an interpretation for our scenario example with domain $\Delta^{\mathcal{I}} = \{x_i \mid 0 \leq i \leq 10\}$, and interpreting the elements of the vocabulary as follows: $\mathsf{Employee}^{\mathcal{I}} = \{x_1, x_2, x_5, x_9\}$, $\mathsf{Company}^{\mathcal{I}} = \{x_6, x_{10}\}$, $\mathsf{Student}^{\mathcal{I}} = \{x_1, x_5, x_7, x_8\}$, $\mathsf{EmpStud}^{\mathcal{I}} = \{x_1, x_5\}$, $\mathsf{Parent}^{\mathcal{I}} = \{x_1, x_2, x_3\}$, $\mathsf{Tax}^{\mathcal{I}} = \{x_4\}$, $\mathsf{pays}^{\mathcal{I}} = \{(x_1, x_0), (x_5, x_4)\}$, $\mathsf{empBy}^{\mathcal{I}} = \{(x_9, x_{10})\}$, $\mathsf{worksFor}^{\mathcal{I}} = \{(x_5, x_6), (x_9, x_{10})\}$, $\mathsf{john}^{\mathcal{I}} = x_5$, $\mathsf{ibm}^{\mathcal{I}} = x_6$, $\mathsf{mary}^{\mathcal{I}} = x_2$.

[1] For the reader not conversant with Description Logics but familiar with modal logics, there are results in the literature relating some families of description logics to systems of modal logic. For example, a well-known result is the one linking the DL $\mathcal{ALC}$ with the normal modal logic K [82].

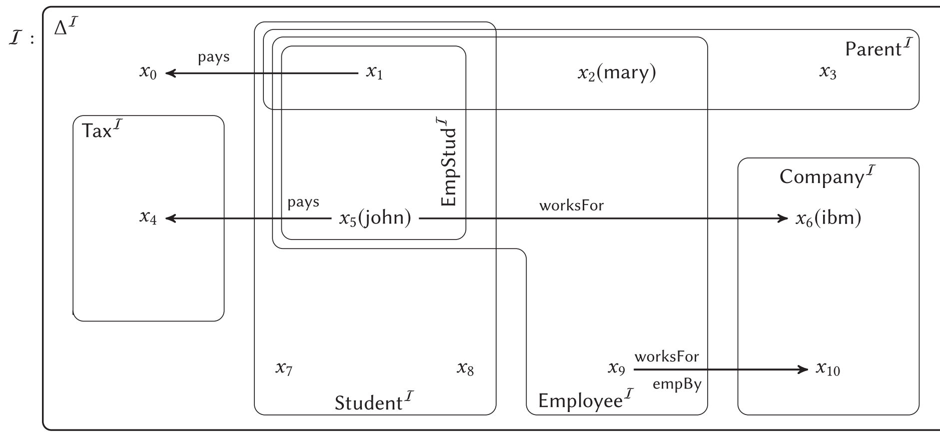

Fig. 1. A DL interpretation.

Let $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ be an interpretation and define $r^{\mathcal{I}}(x) =_{\text{def}} \{y \in \Delta^{\mathcal{I}} \mid (x, y) \in r^{\mathcal{I}}\}$, for $r \in \mathsf{R}$. We extend the interpretation function $\cdot^{\mathcal{I}}$ to interpret complex concepts of $\mathcal{L}$ as follows:

$$\top^{\mathcal{I}} =_{\text{def}} \Delta^{\mathcal{I}};$$

$$\bot^{\mathcal{I}} =_{\text{def}} \emptyset;$$

$$(\neg C)^{\mathcal{I}} =_{\text{def}} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}};$$

$$(C \sqcap D)^{\mathcal{I}} =_{\text{def}} C^{\mathcal{I}} \cap D^{\mathcal{I}};$$

$$(C \sqcup D)^{\mathcal{I}} =_{\text{def}} C^{\mathcal{I}} \cup D^{\mathcal{I}};$$

$$(\exists r.C)^{\mathcal{I}} =_{\text{def}} \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \cap C^{\mathcal{I}} \neq \emptyset\};$$

$$(\forall r.C)^{\mathcal{I}} =_{\text{def}} \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \subseteq C^{\mathcal{I}}\}.$$

For the interpretation $\mathcal{I}$ in Figure 1, we have $(\mathsf{Parent} \sqcap \mathsf{Employee})^{\mathcal{I}} = \{x_1, x_2\}$ and $(\exists \mathsf{pays}.\mathsf{Tax})^{\mathcal{I}} = \{x_5\}$.

Given $C, D \in \mathcal{L}$, a statement of the form $C \sqsubseteq D$ is called a *subsumption statement*, or *general concept inclusion* (GCI), read "C is subsumed by D". A concrete example of GCI is $\mathsf{EmpStud} \sqsubseteq \mathsf{Student} \sqcap \mathsf{Employee}$. $C \equiv D$ is an abbreviation for both $C \sqsubseteq D$ and $D \sqsubseteq C$. An $\mathcal{ALC}$ *TBox* $\mathcal{T}$ is a finite set of GCIs. Given $C \in \mathcal{L}$, $r \in \mathsf{R}$ and $a, b \in \mathsf{I}$, an *assertional statement* (*assertion*, for short) is an expression of the form $a : C$ or $(a, b) : r$, read, respectively, "a is an instance of C" and "a is related to b via r". Examples of assertions are $\mathsf{john} : \mathsf{EmpStud}$ and $(\mathsf{john}, \mathsf{ibm}) : \mathsf{worksFor}$. An $\mathcal{ALC}$ *ABox* $\mathcal{A}$ is a finite set of assertional statements. We denote statements with $\alpha, \beta, \ldots$. Given $\mathcal{T}$ and $\mathcal{A}$, with $\mathcal{KB} =_{\text{def}} \mathcal{T} \cup \mathcal{A}$, we denote an $\mathcal{ALC}$ *knowledge base*, a.k.a., an *ontology*, an example of which follows:

$$\mathcal{T} = \left\{ \begin{array}{c} \mathsf{EmpStud} \sqsubseteq \mathsf{Student} \sqcap \mathsf{Employee}, \\ \mathsf{Student} \sqsubseteq \neg\exists \mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqsubseteq \exists \mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqcap \mathsf{Parent} \sqsubseteq \neg\exists \mathsf{pays}.\mathsf{Tax}, \\ \mathsf{Employee} \sqsubseteq \exists \mathsf{worksFor}.\mathsf{Company} \end{array} \right\},$$

$$\mathcal{A} = \{\mathsf{john} : \mathsf{EmpStud}, \mathsf{john} : \mathsf{Parent}, (\mathsf{john}, \mathsf{ibm}) : \mathsf{worksFor}\}.$$

An interpretation $\mathcal{I}$ *satisfies* a GCI $C \sqsubseteq D$ (denoted $\mathcal{I} \Vdash C \sqsubseteq D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. (And then $\mathcal{I} \Vdash C \equiv D$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$.) $\mathcal{I}$ *satisfies* an assertion $a : C$ (respectively, $(a, b) : r$), denoted $\mathcal{I} \Vdash a : C$ (respectively, $\mathcal{I} \Vdash (a, b) : r$), if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (respectively, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$).

In the interpretation $\mathcal{I}$ in Figure 1, we have $\mathcal{I} \Vdash \mathsf{EmpStud} \sqsubseteq \mathsf{Student} \sqcap \mathsf{Employee}$ and $\mathcal{I} \Vdash \mathsf{john} : \exists \mathsf{pays}.\mathsf{Tax}$, but we do not have $\mathcal{I} \Vdash (\mathsf{john}, \mathsf{ibm}) : \mathsf{empBy}$.

We say that an interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ (respectively, of an ABox $\mathcal{A}$), denoted $\mathcal{I} \Vdash \mathcal{T}$ (respectively, $\mathcal{I} \Vdash \mathcal{A}$) if $\mathcal{I} \Vdash \alpha$ for every α in $\mathcal{T}$ (respectively, in $\mathcal{A}$). We say that $\mathcal{I}$ is a model of a knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{A}$ if $\mathcal{I} \Vdash \mathcal{T}$ and $\mathcal{I} \Vdash \mathcal{A}$. It can be verified that the interpretation in Figure 1 is not a model of the example knowledge base above. (Actually, it is not hard to see that the knowledge base above admits no model.)

A statement α is (classically) *entailed* by a knowledge base $\mathcal{KB}$, denoted $\mathcal{KB} \models \alpha$, if every model of $\mathcal{KB}$ satisfies α. If $\mathcal{I} \Vdash \alpha$ for all interpretations $\mathcal{I}$, then we say α is a *validity* and denote this fact with $\models \alpha$.

The focus of the present article being on defeasibility for description logic TBoxes only, we henceforth assume the ABox is empty. (We are currently in the process of extending our approach to description logic knowledge bases, with ABoxes included into the mix.) It is easy to see that, for $\mathcal{T}$ as above, we have $\mathcal{T} \models \mathsf{EmpStud} \sqsubseteq \bot$.

For more details on Description Logics in general and on $\mathcal{ALC}$ in particular, the reader is invited to consult the Description Logic Handbook [1] and the introductory textbook on Description Logic [4].

3 FOUNDATIONS FOR DEFEASIBILITY IN DLS

In this section, we lay the formal foundations of our approach to defeasible reasoning in DL ontologies. For the most part, we build on the so-called preferential approach to non-monotonic reasoning [67, 70, 85].

3.1 Defeasible Subsumption Relations and their KLM-style Properties

In a sense, class subsumption (alias concept inclusion) of the form $C \sqsubseteq D$ is the main notion in DL ontologies. Given its implication-like intuition, subsumption lends itself naturally to defeasibility: "provisionally, if an object falls under C, then it also falls under D", as in "usually, students are tax exempted". In that respect, a defeasible version of concept inclusion is the starting point for an investigation of defeasible reasoning in DL ontologies. (We shall also address defeasibility of the entailment relation in later sections.)

Definition 1 (Defeasible Concept Inclusion). Let $C, D \in \mathcal{L}$. A defeasible concept inclusion axiom (DCI, for short) is a statement of the form $C \sqsubset\!\!\sim D$.

A defeasible concept inclusion of the form $C \sqsubset\!\!\sim D$ is to be read as "*usually*, an instance of the class C is also an instance of the class D". For instance, the DCI $\mathsf{Student} \sqsubset\!\!\sim \neg\exists \mathsf{pays}.\mathsf{Tax}$ formalises the example above. Paraphrasing Lehmann [68], the intuition of $C \sqsubset\!\!\sim D$ is that "if [the fact it belongs to] C were all the information about an object available to an agent, then [that it also belongs to] D would be a sensible conclusion to draw about such an object". It is worth noting that $\sqsubset\!\!\sim$, just as $\sqsubseteq$, is a "connective" sitting between the concept language (object level) and the meta-language (that of entailment) and it is meant to be the defeasible counterpart of the classical subsumption $\sqsubseteq$.

Being (defeasible) statements, DCIs will also be denoted by $\alpha, \beta, \ldots$ Whenever a distinction between GCIs and DCIs is in order, we shall make it explicitly.

Definition 2 (Defeasible TBox). A defeasible TBox (DTBox, for short) is a finite set of DCIs.

Given a TBox $\mathcal{T}$ and a DTBox $\mathcal{D}$, we let $\mathcal{KB} =_{\text{def}} \mathcal{T} \cup \mathcal{D}$ and refer to it as a *defeasible knowledge base* (alias *defeasible ontology*).

Example 1. The following defeasible knowledge base gives a formal specification for our student scenario:

$$\mathcal{T} = \{\mathsf{EmpStud} \sqsubseteq \mathsf{Student}\},$$

$$\mathcal{D} = \left\{ \begin{array}{c} \mathsf{Student} \sqsubsetsim \neg\exists \mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqsubsetsim \exists \mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqcap \mathsf{Parent} \sqsubsetsim \neg\exists \mathsf{pays}.\mathsf{Tax} \end{array} \right\}.$$

In our semantic construction later on in Section 4.1, it will also be useful to be able to refer to infinite sets of concept inclusions. Let $\mathcal{K}$ therefore denote a *defeasible theory*, defined as a defeasible knowledge base but without the restriction on $\mathcal{T}$ and $\mathcal{D}$ to finite sets.

To assess the behaviour of the new connective and check it against both the intuition and the set of properties usually considered in a non-monotonic setting, it is convenient to look at a set of $\sqsubsetsim$-statements as a binary relation of the "antecedent-consequent" kind.

Definition 3 (Defeasible Subsumption Relation). A defeasible subsumption relation is a binary relation $\sqsubsetsim \subseteq \mathcal{L} \times \mathcal{L}$.

The idea is to mimic the analysis of defeasible entailment relations carried out by Kraus et al. [67] in the propositional case, where entailment is seen as a binary relation on the set of propositional sentences. Here, we shall adopt the view of subsumption as a binary relation on concepts of our description language.

Sometimes (e.g., in the structural properties below), we shall write $(C, D) \in \sqsubsetsim$ in the infix notation, i.e., as $C \sqsubsetsim D$. The context will make clear when we will be talking about elements of a relation or statements (DCIs) in a defeasible knowledge base. Whenever disambiguation is in order, we shall flag it to the reader.

Definition 4 (Preferential Subsumption Relation). A defeasible subsumption relation $\sqsubsetsim$ is a **preferential subsumption relation** if it satisfies the following set of properties, which we refer to as the (DL versions of the) preferential KLM properties:

$$\text{(Cons)}\ \top \not\sqsubsetsim \bot, \qquad \text{(Ref)}\ C \sqsubsetsim C, \qquad \text{(LLE)}\ \frac{C \equiv D,\ C \sqsubsetsim E}{D \sqsubsetsim E},$$

$$\text{(And)}\ \frac{C \sqsubsetsim D,\ C \sqsubsetsim E}{C \sqsubsetsim D \sqcap E}, \quad \text{(Or)}\ \frac{C \sqsubsetsim E,\ D \sqsubsetsim E}{C \sqcup D \sqsubsetsim E}, \quad \text{(RW)}\ \frac{C \sqsubsetsim D,\ D \sqsubseteq E}{C \sqsubsetsim E},$$

$$\text{(CM)}\ \frac{C \sqsubsetsim D,\ C \sqsubsetsim E}{C \sqcap D \sqsubsetsim E}.$$

The (Cons) property is a consequence of the adoption of a DL-based semantics, which enforces the non-emptiness of the domain, as will become clear in the next section. The rest of the properties in Definition 4 result from a translation of the properties for preferential consequence relations proposed by Kraus et al. [67] in the propositional setting. They have been discussed at length in the literature for both the propositional and the DL cases [27, 30, 56, 57, 67, 70], and we shall not repeat so here.

If, in addition to the preferential properties above, the relation $\precsim$ also satisfies rational monotonicity (RM) below, then it is said to be a *rational* subsumption relation:

$$\text{(RM)}\ \frac{C \precsim D,\ C \not\precsim \neg E}{C \sqcap E \precsim D}.$$

Rational monotonicity is often considered a desirable property to have, one of the reasons stemming from the fact it is a necessary condition for the satisfaction of the principle of *presumption of typicality* [69, Section 3.1]. Such a principle is a simple yet intuitive formalisation of a form of reasoning we carry out when facing lack of information: We reason assuming that we are in the most typical possible situation, compatible with the information at our disposal. (More details will be provided in Section 4).

3.2 Preferential Semantics and Representation Results

In this section, we present our semantics for preferential and rational subsumption by enriching standard DL interpretations $\mathcal{I}$ with an ordering on the elements of the domain $\Delta^{\mathcal{I}}$. The intuition underlying this is simple and natural, and extends similar work done for the propositional case by Shoham [85], Kraus et al. [67], Lehmann and Magidor [70], and Booth et al. [13–16] to the case for description logics. This is not the first extension of this kind, as evidenced by the work of Boutilier [18], Baltag and Smets [5, 6], Giordano et al. [54, 56–60], Britz et al. [25–28, 30], and Britz and Varzinczak [32, 33, 35–38]. However, this is the first comprehensive semantic account of both preferential and rational subsumption relations, with accompanying representation results, based on the standard semantics for description logics.

Informally, our semantic constructions are based on the idea that objects of the domain can be ordered according to their degree of *normality* [18] or *typicality* [15, 16, 28, 54]. Paraphrasing Boutilier [18, pp. 110–116],

> Surely, there is no *inherent* property of objects that allows them to be judged to be more or less normal in absolute terms. These orderings are purely "subjective" (in the sense that they can be thought of as part of an agent's belief state) and the space of orderings deemed plausible by the agent may (among other things) be determined by, e.g., empirical data. By using orderings in this way, we encode our (or the agent's) *expectations* about the objects corresponding to their perceived regularity or typicality. Those objects not violating our expectations are considered to be more normal than the objects that violate some.

Hence, we do not require that there exists something intrinsic about objects that makes one object more normal than another. Rather, the intention is to provide a framework in which to express all conceivable ways in which objects, with their associated properties and relationships with other objects, can be ordered in terms of typicality, in the same way that the class of all standard DL interpretations constitute a framework representing all conceivable ways of representing the properties of objects and their relationships with other objects. Just as the latter are constrained by stating subsumption statements in a knowledge base, the possible orderings that are considered plausible are encoded by writing down DCIs.

That said, we are ready for the definition of the first semantic construction the present work relies on.

Definition 5 (Preferential Interpretation). A **preferential interpretation** is a tuple $\mathcal{P} =_{\text{def}} \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec^{\mathcal{P}} \rangle$, where $\langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}} \rangle$ is a (standard) DL interpretation (which we denote by $\mathcal{I}_{\mathcal{P}}$ and refer to as the classical interpretation associated with $\mathcal{P}$), and $\prec^{\mathcal{P}}$ is a strict partial order on $\Delta^{\mathcal{P}}$ (i.e.,

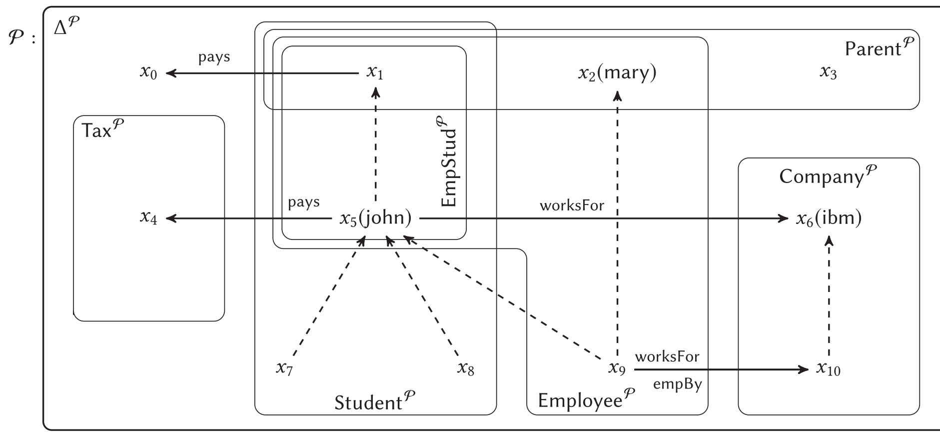

Fig. 2. A preferential interpretation.

$\prec^{\mathcal{P}}$ is irreflexive and transitive) satisfying the smoothness condition (for every $C \in \mathcal{L}$, if $C^{\mathcal{P}} \neq \emptyset$, then $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \neq \emptyset$).[2]

A first version of this definition was first proposed by Giordano et al. [54], and was followed shortly thereafter by a similar definition, courtesy of Britz et al. [27].

Figure 2 depicts a preferential interpretation in our scenario example where $\Delta^{\mathcal{P}}$ and $\cdot^{\mathcal{P}}$ are as in the interpretation $\mathcal{I}$ shown in Figure 1, and $\prec^{\mathcal{P}} = \{(x_7, x_5), (x_8, x_5), (x_9, x_5), (x_5, x_1), (x_7, x_1), (x_8, x_1), (x_9, x_1), (x_9, x_2), (x_{10}, x_6)\}$, represented by the dashed arrows in the picture. (For the sake of presentation, in the picture we omit the transitive $\prec^{\mathcal{P}}$-arrows.)

Preferential interpretations provide us with a simple and intuitive way to give a semantics to DCIs.

Definition 6 (Satisfaction). Let $\mathcal{P}$ be a preferential interpretation, $C, D \in \mathcal{L}$, $r \in \mathsf{R}$ and $a, b \in \mathsf{I}$. The **satisfaction relation** $\Vdash$ is defined as follows:

- $\mathcal{P} \Vdash C \sqsubseteq D$ if $C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$;
- $\mathcal{P} \Vdash C \mathrel{\sqsubseteq\!\!\!\!\raisebox{-0.6ex}{$\sim$}} D$ if $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$.

If $\mathcal{P} \Vdash \alpha$, then we say $\mathcal{P}$ **satisfies** α. $\mathcal{P}$ satisfies a defeasible knowledge base $\mathcal{KB}$, written $\mathcal{P} \Vdash \mathcal{KB}$, if $\mathcal{P} \Vdash \alpha$ for every $\alpha \in \mathcal{KB}$, in which case we say $\mathcal{P}$ is a **preferential model** of $\mathcal{KB}$. We say $C \in \mathcal{L}$ is **satisfiable** w.r.t. $\mathcal{KB}$ if there is a model $\mathcal{P}$ of $\mathcal{KB}$ s.t. $C^{\mathcal{P}} \neq \emptyset$.

The semantics proposed above was suggested by Giordano et al. [54] in their definition of a typicality operator, and by Britz et al. [27] as above. The intuition underlying the definition of satisfaction for $C \mathrel{\sqsubseteq\!\!\!\!\raisebox{-0.6ex}{$\sim$}} D$ above should be clear. It states that, in the most general case, any instance of C should also be an instance of D. It is, of course, possible to consider other definitions of $C \mathrel{\sqsubseteq\!\!\!\!\raisebox{-0.6ex}{$\sim$}} D$, such as that all typical instances of C should be typical instances of D. However, our definition can be justified by pointing out that it is a version of the definition for defeasible consequence (denoted by $\mid\!\sim$) by Kraus et al. [67], adjusted to apply to defeasible inclusion for description logics. The origins of the definition of $\mid\!\sim$ can be traced back to the work of Shoham [85] and is a cornerstone of the preferential approach to defeasible reasoning.

It is easy to see that the addition of the $\prec^{\mathcal{P}}$-component preserves the truth of all classical subsumption statements holding in the remaining structure:

[2]Given $X \subseteq \Delta^{\mathcal{P}}$, with $\min_{\prec^{\mathcal{P}}} X$, we denote the set $\{x \in X \mid \text{for every } y \in X, (y, x) \notin \prec^{\mathcal{P}}\}$.

Lemma 1. *Let $\mathcal{P}$ be a preferential interpretation. For every $C, D \in \mathcal{L}$, $\mathcal{P} \Vdash C \sqsubseteq D$ if and only if $\mathcal{I}_{\mathcal{P}} \Vdash C \sqsubseteq D$.*

It is worth noting that, due to smoothness of $\prec^{\mathcal{P}}$, every (classical) subsumption statement is equivalent, with respect to preferential interpretations, to some DCI.

Lemma 2. *For every preferential interpretation $\mathcal{P}$, and every $C, D \in \mathcal{L}$, $\mathcal{P} \Vdash C \sqsubseteq D$ if and only if $\mathcal{P} \Vdash C \sqcap \neg D \precsim \bot$.*

The following result, of which the proof can be found in Appendix A, will come in handy later on.

Lemma 3. *Preferential interpretations are closed under disjoint union.*

An obvious question that can now be raised is: "How do we know our preferential semantics provides an appropriate meaning to the notion of defeasible concept inclusion?" The following definition will help us in answering this question:

Definition 7 ($\mathcal{P}$-Induced Defeasible Subsumption). Let $\mathcal{P}$ be a preferential interpretation. Then $\precsim_{\mathcal{P}} =_{\text{def}} \{(C, D) \mid \mathcal{P} \Vdash C \precsim D\}$ is the **defeasible subsumption relation induced by** $\mathcal{P}$.

The first important result we present here, which also answers the above raised question, shows that there is a full correspondence between the class of preferential subsumption relations and the class of defeasible subsumption relations induced by preferential interpretations. It is the DL analogue of a representation result proved by Kraus et al. for the propositional case [67, Theorem 3] and its proof can be found in Appendix B.

Theorem 1 (Representation Result for Preferential Subsumption). *A defeasible subsumption relation $\precsim \subseteq \mathcal{L} \times \mathcal{L}$ is preferential if and only if there is a preferential interpretation $\mathcal{P}$ such that $\precsim_{\mathcal{P}} = \precsim$.*

What is perhaps surprising about this result is that no additional properties based on the syntactic structure of the underlying DL are necessary to characterise the defeasible subsumption relations induced by preferential interpretations. We provide below a few properties involving the use of quantifiers that are satisfied by all preferential subsumption relations. (See Section 5 for more on properties explicitly mentioning DL-specific constructs.)

The first two are "existential" and "universal" versions of cautious monotonicity (CM):

$$(\text{CM}_\exists)\ \frac{\exists r.C \precsim E,\ \exists r.C \precsim \forall r.D}{\exists r.(C \sqcap D) \precsim E},$$

$$(\text{CM}_\forall)\ \frac{\forall r.C \precsim E,\ \forall r.C \precsim \forall r.D}{\forall r.(C \sqcap D) \precsim E}.$$

The third one is a rephrasing of the Rule of Necessitation in modal logic [50]. It guarantees the absence of so-called *spurious objects* [31] in the original preferential semantics for DLs by Britz et al. [29, 30]. That is, if C is unsatisfiable, then so is $\exists r.C$ (cf. Lemma 2):

$$(\text{Norm})\ \frac{C \precsim \bot}{\exists r.C \precsim \bot}.$$

In addition to preferential interpretations, we are also interested in the study of *modular* interpretations, which are preferential interpretations in which the $\prec$-component is a *modular* ordering:

Definition 8 (Modular Order). Given a set X, $\prec \subseteq X \times X$ is **modular** if it is a strict partial order, and its associated incomparability relation $\sim$, defined by $x \sim y$ if neither $x \prec y$ nor $y \prec x$, is transitive.

If $\prec$ is modular, then $\sim$ is an equivalence relation.

Definition 9 (Modular Interpretation). A **modular interpretation** is a preferential interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ such that $\prec^{\mathcal{R}}$ is modular.

Modular interpretations (albeit under the name *ranked interpretations*) were first inroduced by Britz et al. [27]. Intuitively, modular interpretations allow us to compare any two objects w.r.t. their plausibility. Those that are incomparable are viewed as being equally plausible. As such, modular interpretations are special cases of preferential interpretations, where plausibility can be represented by any smooth strict partial order.

The main reason to consider modular interpretations is that they provide the semantic foundation of rational subsumption relations. This is made precise by our second important result below, which shows that the defeasible subsumption relations induced by modular interpretations are precisely the rational subsumption relations. Again, this is the DL analogue of a representation result proved by Lehmann and Magidor for the propositional case [70, Theorem 5] and its proof can be found in Appendix C.

Theorem 2 (Representation Result for Rational Subsumption). *A defeasible subsumption relation $\sqsubsim \subseteq \mathcal{L} \times \mathcal{L}$ is rational if and only if there is a modular interpretation $\mathcal{R}$ such that $\sqsubsim_{\mathcal{R}} = \sqsubsim$.*

Analogous to the case for cautious monotonicity above, the following "existential" and "universal" versions of rational monotonicity are satisfied by all rational subsumption relations:

$$(\mathrm{RM}_{\exists})\ \frac{\exists r.C \sqsubsim E,\ \exists r.C \not\sqsubsim \forall r.\neg D}{\exists r.(C \sqcap D) \sqsubsim E},$$

$$(\mathrm{RM}_{\forall})\ \frac{\forall r.C \sqsubsim E,\ \forall r.C \not\sqsubsim \forall r.\neg D}{\forall r.(C \sqcap D) \sqsubsim E}.$$

It is worth pausing for a moment to emphasise the significance of these two results (Theorems 1 and 2). They provide exact semantic characterisations of two important classes of defeasible subsumption relations, namely, preferential and rational subsumption, in terms of the classes of preferential and modular interpretations, respectively. As we shall see in Section 4, these results form the core of the investigation into an appropriate notion of entailment for defeasible DL ontologies.

4 RATIONALITY IN ENTAILMENT

From the standpoint of knowledge representation and reasoning, a pivotal question is that of deciding which statements are *entailed* by a knowledge base. We shall devote the remainder of the article to this matter, and in this section we lay out the formal foundations for that.

4.1 Preferential Entailment

In the exploration of a notion of entailment for defeasible ontologies, an obvious starting point is to consider a Tarskian definition of consequence:

Definition 10 (Preferential Entailment). A statement α is **preferentially entailed** by a defeasible knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\mathrm{pref}} \alpha$, if every preferential model of $\mathcal{KB}$ satisfies α.

As usual, this form of entailment is accompanied by a corresponding notion of closure.

Definition 11 (Preferential Closure). Let $\mathcal{KB}$ be a defeasible knowledge base. With $\mathcal{KB}^{*}_{\mathrm{pref}} =_{\mathrm{def}} \{\alpha \mid \mathcal{KB} \models_{\mathrm{pref}} \alpha\}$, we denote the **preferential closure** of $\mathcal{KB}$.

The terms "preferential entailment" and "preferential closure" were first applied by Lehmann and Magidor in the propositional context [70]. Intuitively, the preferential closure of a defeasible knowledge base $\mathcal{KB}$ corresponds to the "core" set of statements, classical and defeasible, that should hold given those in $\mathcal{KB}$. Hence, preferential entailment and preferential closure are two sides of the same coin, mimicking an analogous result for preferential reasoning in the propositional [67] case.

Recall (cf. the discussion following Definition 2) that a defeasible theory $\mathcal{K}$ is a defeasible knowledge base without the restriction to finite sets. When assessing how appropriate a notion of entailment for defeasible ontologies is, the following definitions turn out to be useful, as will become clear in the sequel:

Definition 12 ($\mathcal{K}$-Induced Defeasible Subsumption). Let $\mathcal{K}$ be a defeasible theory. Then (1) $\mathcal{D}_{\mathcal{K}} =_{\text{def}} \{C \precsim D \mid C \precsim D \in \mathcal{K}\} \cup \{C \sqcap \neg D \precsim \bot \mid C \sqsubseteq D \in \mathcal{K}\}$ is the **DTBox induced by** $\mathcal{K}$ and (2) $\precsim_{\mathcal{K}} =_{\text{def}} \{(C, D) \mid C \precsim D \in \mathcal{D}_{\mathcal{K}}\}$ is the **defeasible subsumption relation induced by** $\mathcal{K}$.

To see why $C \sqcap \neg D \precsim \bot$ is an appropriate representation of $C \sqsubseteq D$ in the definition above, recall from Lemma 2 that $C \sqcap \neg D \precsim \bot$ is equivalent to $C \sqsubseteq D$ on the level of preferential interpretations.

So, the DTBox induced by $\mathcal{K}$ is the set of defeasible subsumption statements contained in $\mathcal{K}$, together with the defeasible versions of the classical subsumption statements in $\mathcal{K}$. The defeasible subsumption relation induced by $\mathcal{K}$ is simply the defeasible subsumption relation corresponding to $\mathcal{D}_{\mathcal{K}}$.

Definition 13. A defeasible theory $\mathcal{K}$ is called **preferential** if the subsumption relation induced by it satisfies the preferential properties in Definition 4.

It turns out that the defeasible subsumption relation induced by the preferential closure of a defeasible knowledge base $\mathcal{KB}$ is exactly the intersection of the defeasible subsumption relations induced by the preferential defeasible theories containing $\mathcal{KB}$.

LEMMA 4. *Lemma Let $\mathcal{KB}$ be a defeasible knowledge base. Then,*

$$\precsim_{\mathcal{KB}^*_{\text{pref}}} = \bigcap\{\precsim_{\mathcal{K}} \mid \mathcal{KB} \subseteq \mathcal{K} \textit{ and } \mathcal{K} \textit{ is preferential}\}.$$

It follows immediately that the preferential closure of a defeasible knowledge base $\mathcal{KB}$ is preferential, and induces the smallest defeasible subsumption relation induced by a preferential defeasible theory containing $\mathcal{KB}$.

Preferential entailment is not always desirable, one of the reasons being that it is monotonic, courtesy of the Tarskian notion of consequence it relies on (see Definition 10). In most cases, as witnessed by the great deal of work in the non-monotonic reasoning community, a move towards rationality is in order. Thanks to the definitions above and the result in Theorem 2, we already know where to start looking for it.

Definition 14 (Modular Entailment). A statement α is **modularly entailed** by a defeasible knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\text{mod}} \alpha$, if every modular model of $\mathcal{KB}$ satisfies α.

As is the case for preferential entailment, modular entailment is accompanied by a corresponding notion of closure.

Definition 15 (Modular Closure). Let $\mathcal{KB}$ be a defeasible knowledge base. With

$$\mathcal{KB}^*_{\text{mod}} =_{\text{def}} \{\alpha \mid \mathcal{KB} \models_{\text{mod}} \alpha\},$$

we denote the **modular closure** of $\mathcal{KB}$.

Definition 16. A defeasible theory $\mathcal{K}$ is called **rational** if it is preferential and $\precsim_{\mathcal{K}}$ is also closed under the rational monotonicity rule (RM).

For modular closure, we get a result similar to Lemma 4.

Lemma 5. *Let $\mathcal{KB}$ be a defeasible knowledge base. Then,*

$$\precsim_{\mathcal{KB}^*_{\mathrm{mod}}} = \bigcap \{ \precsim_{\mathcal{K}} \mid \mathcal{KB} \subseteq \mathcal{K} \text{ and } \mathcal{K} \text{ is rational} \}.$$

That is, the modular closure of a defeasible knowledge base $\mathcal{KB}$ induces the smallest defeasible subsumption relation induced by a rational defeasible theory containing $\mathcal{KB}$. However, the modular closure of a defeasible knowledge base $\mathcal{KB}$ is not necessarily rational. That is, if one looks at the set of statements (in particular the $\precsim$-ones) modularly entailed by a knowledge base as a defeasible subsumption relation, then it need not satisfy the rational monotonicity property. Even worse, modular entailment *coincides* with preferential entailment, as the following result, adapted from a well-known similar result in the propositional case [70, Theorem 4.2], and originally adapted for description logics by Giordano et al. [62], shows.

Lemma 6. $\mathcal{KB}^*_{\mathrm{mod}} = \mathcal{KB}^*_{\mathrm{pref}}$.

As a result, modular entailment unfortunately falls short of providing us with an appropriate notion of non-monotonic entailment. To see why this result is problematic, note firstly that modular entailment is, by definition, a *monotonic* form of entailment. That is, once we have concluded that an inclusion, whether classical or defeasible, is modularly entailed by a knowledge base $\mathcal{KB}$, that inclusion cannot be retracted if additional statements are added to $\mathcal{KB}$. This leads, interestingly enough, to modular entailment being too conservative in terms of what can be entailed from it, as the following example illustrates.

Example 2. Consider the following defeasible knowledge base:

$$\mathcal{T} = \{\mathsf{Penguin} \sqsubseteq \mathsf{Bird}\},$$

$$\mathcal{D} = \{\mathsf{Bird} \precsim \mathsf{Flies}, \mathsf{Bird} \precsim \mathsf{HasWings}\}.$$

Modular entailment does not allow us to draw the reasonable conclusion that penguins usually (or provisionally) have wings ($\mathsf{Penguin} \precsim \mathsf{HasWings}$). This is because, on subsequently learning that penguins usually do not have wings ($\mathsf{Penguin} \precsim \neg\mathsf{HasWings}$) and adding that to our knowledge base, the monotonicity of modular entailment would force us retain the conclusion that penguins usually have wings ($\mathsf{Penguin} \precsim \mathsf{HasWings}$), in addition to our newfound knowledge that penguins usually do not have wings. This, in turn, would allow us to conclude that there are not any penguins ($\mathsf{Penguin} \precsim \bot$).

It is because of Lemma 6 and consequences such as those embodied in Example 2 that Lehmann and Magidor insisted on the requirement that entailment w.r.t. defeasible knowledge bases be *rational* in the sense of satisfying the rational monotonicity property [70]. This can rightly be regarded as one of the basic tenets of this approach to entailment. In what follows, we address precisely this issue.

4.2 Semantic Rational Entailment

In this section, we introduce a definition of semantic entailment, which, as we shall see, is appropriate in the light of the discussion above. The constructions we are going to present are inspired by the semantic characterisation of rational closure by Booth and Paris [17] in the propositional case. We shall give a corresponding proof-theoretic characterisation of our version of semantic entailment in Section 5.1.

We focus our attention on a subclass of modular orders, referred to as *ranked orders*:

Definition 17 (Ranked Order). Given a set X, the binary relation $\prec \subseteq X \times X$ is a **ranked order** if there is a mapping $h_{\mathcal{R}} : X \longrightarrow \mathbb{N}$ satisfying the following convexity property:

- for every $i \in \mathbb{N}$, if for some $x \in X$ $h_{\mathcal{R}}(x) = i$, then, for every j such that $0 \leq j < i$, there is a $y \in X$ for which $h_{\mathcal{R}}(y) = j$,

and s.t. for every $x, y \in X$, $x \prec y$ iff $h_{\mathcal{R}}(x) < h_{\mathcal{R}}(y)$.

It is easy to see that a ranked order $\prec$ is also modular: $\prec$ is a strict partial order, and, since two objects x, y are incomparable (i.e., $x \sim y$) if and only if $h_{\mathcal{R}}(x) = h_{\mathcal{R}}(y)$, $\sim$ is a transitive relation. By constraining our preference relations to the ranked orders, we can identify a subset of the modular interpretations we refer to as the *ranked interpretations*.

Definition 18 (Ranked Interpretation). A **ranked interpretation** is a modular interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ s.t. $\prec^{\mathcal{R}}$ is a ranked order.

We now provide two basic results about ranked interpretations. First, all finite modular interpretations are ranked interpretations.

Lemma 7. *A modular interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ s.t. $\Delta^{\mathcal{R}}$ is finite is a ranked interpretation.*

Next, for every ranked interpretation $\mathcal{R}$, the function $h_{\mathcal{R}}(\cdot)$ is unique.

Proposition 1. *Given a ranked interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, there is only one function $h_{\mathcal{R}} : X \longrightarrow \mathbb{N}$ satisfying the convexity property and s.t. for every $x, y \in X$, $x \prec y$ iff $h_{\mathcal{R}}(x) < h_{\mathcal{R}}(y)$.*

Proposition 1 allows us to use the function $h_{\mathcal{R}}(\cdot)$ to define the notions of *height* and *layers*.

Definition 19 (Height & Layers). Given a ranked interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, its characteristic ranking function $h_{\mathcal{R}}(\cdot)$, and an object $x \in \Delta^{\mathcal{R}}$, $h_{\mathcal{R}}(x)$ is called the **height** of x in $\mathcal{R}$.

For every ranked interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, we can partition the domain $\Delta^{\mathcal{R}}$ into a sequence of *layers* $(L_0, \ldots, L_n, \ldots)$, where, for every object $x \in \Delta^{\mathcal{R}}$, we have $x \in L_i$ iff $h_{\mathcal{R}}(x) = i$.

Intuitively, the lower the height of an object in an interpretation $\mathcal{R}$, the more typical (or normal) the object is in $\mathcal{R}$. We can also think of a level of typicality for concepts: the height of a concept $C \in \mathcal{L}$ in $\mathcal{R}$ is the index of the layer to which the restriction of the concept's extension to its $\prec^{\mathcal{R}}$-minimal elements belong, i.e., $h_{\mathcal{R}}(C) = i$ if $\emptyset \subset \min_{\prec^{\mathcal{R}}} C^{\mathcal{R}} \subseteq L_i$. As a convention, if $\min_{\prec^{\mathcal{R}}} C^{\mathcal{R}} = \emptyset$, that is, if $C^{\mathcal{R}} = \emptyset$, then $h_{\mathcal{R}}(C) = \infty$.

The following result (stated by Giordano et al. [60] as a consequence of their results, and proved directly in Appendix D), will be useful for some of the proofs in later sections of the article:

Theorem 3 (Finite-Model Property). *Defeasible $\mathcal{ALC}$ has the finite-model property. In particular, every defeasible $\mathcal{ALC}$ knowledge base that has a modular model, has also a finite-ranked model.*

Given a set of ranked interpretations, we can introduce a new form of model merging, *ranked union*.

Definition 20 (Ranked Union). Given a countable set of ranked interpretations $\mathfrak{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$, a ranked interpretation $\mathcal{R}^{\mathfrak{R}} =_{\text{def}} \langle \Delta^{\mathfrak{R}}, \cdot^{\mathfrak{R}}, \prec^{\mathfrak{R}} \rangle$ is the **ranked union** of $\mathfrak{R}$ if the following holds:

- $\Delta^{\mathfrak{R}} =_{\text{def}} \coprod_{\mathcal{R} \in \mathfrak{R}} \Delta^{\mathcal{R}}$, i.e., the disjoint union of the domains from $\mathfrak{R}$, where each $\mathcal{R} \in \mathfrak{R}$ has the elements $x, y, \ldots$ of its domain renamed as $x_{\mathcal{R}}, y_{\mathcal{R}}, \ldots$ so that they are all distinct in $\Delta^{\mathfrak{R}}$;
- $x_{\mathcal{R}} \in A^{\mathfrak{R}}$ iff $x \in A^{\mathcal{R}}$;
- $(x_{\mathcal{R}}, y_{\mathcal{R}'}) \in r^{\mathfrak{R}}$ iff $\mathcal{R} = \mathcal{R}'$ and $(x, y) \in r^{\mathcal{R}}$;
- for every $x_{\mathcal{R}} \in \Delta^{\mathfrak{R}}$, $h_{\mathfrak{R}}(x_{\mathcal{R}}) = h_{\mathcal{R}}(x)$.

The latter condition corresponds to imposing that $x_{\mathcal{R}} \prec^{\Re} y_{\mathcal{R}'}$ iff $h_{\mathcal{R}}(x) < h_{\mathcal{R}'}(y)$.

Informally, the ranked union of a set of ranked interpretations is the result of merging all their layers of height i into a single layer of height i, for all i.

Lemma 8. *Given a set of ranked models of a defeasible knowledge base $\mathcal{KB}$, their ranked union is itself a ranked model of $\mathcal{KB}$.*

Let $\mathcal{KB}$ be a defeasible knowledge base and let Δ be a fixed countably infinite set. Define

$$Mod_{\Delta}(\mathcal{KB}) =_{\text{def}} \{\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle \mid \mathcal{R} \Vdash \mathcal{KB}, \mathcal{R} \text{ is ranked and } \Delta^{\mathcal{R}} = \Delta\}.$$

The following result shows that the set $Mod_{\Delta}(\mathcal{KB})$ suffices to characterise modular entailment (the proof is in Appendix D):

Lemma 9. *For every $\mathcal{KB}$ and every $C, D \in \mathcal{L}$, $\mathcal{KB} \models_{\text{mod}} C \precsim D$ iff $\mathcal{R} \Vdash C \precsim D$, for every $\mathcal{R} \in Mod_{\Delta}(\mathcal{KB})$.*

Therefore, we can use just the set of interpretations in $Mod_{\Delta}(\mathcal{KB})$ to decide the consequences of $\mathcal{KB}$ w.r.t. modular entailment.

We can now use the set $Mod_{\Delta}(\mathcal{KB})$ as a springboard to introduce what will turn out to be a canonical modular interpretation for $\mathcal{KB}$. Using $Mod_{\Delta}(\mathcal{KB})$ and ranked union, we can define the following relevant model.

Definition 21 (Big Ranked Model). Let $\mathcal{KB}$ be a defeasible knowledge base. The **big ranked model** of $\mathcal{KB}$ is the ranked model $\mathcal{O} =_{\text{def}} \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^{\mathcal{O}} \rangle$ that is the ranked union of the models in $Mod_{\Delta}(\mathcal{KB})$.

Given Lemma 8, we can state the following:

Corollary 1. *$\mathcal{O}$ is a ranked model of $\mathcal{KB}$.*

Armed with the definitions and results above, we are now ready to provide an alternative definition of entailment in the context of defeasible ontologies:

Definition 22 (Rational Entailment). A statement α is **rationally entailed** by a knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\text{rat}} \alpha$, if $\mathcal{O} \Vdash \alpha$.

Giordano and colleagues [60, 61] provide a different semantics in which some modular interpretations are preferred over others. It has been shown that our definition corresponds to the main semantic construction they propose [49, Proposition 30].

That our notion of entailment indeed deserves its name is witnessed by the following result, a consequence of Corollary 1 and Theorem 2:

Corollary 2. *Let $\mathcal{KB}$ be a defeasible knowledge base. $\{C \precsim D \mid \mathcal{O} \Vdash C \precsim D\}$ is rational.*

In conclusion, rational entailment is a good candidate for the appropriate notion of defeasible consequence we have been looking for. Of course, a question that arises is whether a notion of closure, in the spirit of preferential and modular closures, that is equivalent to it can be defined. In the next section, we address precisely this matter.

5 RATIONAL CLOSURE FOR DEFEASIBLE KNOWLEDGE BASES

We now turn our attention to the exploration, in a DL setting, of the well-known notion of *rational closure* of a defeasible knowledge base as studied by Lehmann and Magidor [70] for propositional logic. For the most part, we base our constructions on the work by Casini and Straccia [45, 48], amending it wherever necessary. As we shall see, rational closure provides a proof-theoretic characterisation of rational entailment and the complexity of its computation is no higher than that of computing entailment in the underlying classical DL. As mentioned in the previous section, an alternative semantic characterisation of rational closure in DLs, one that is equivalent to ours, has been proposed by Giordano et al. [60, 61]. They also define an alternative method for computing rational closure [60, 62]. This is elaborated on in Section 6.

5.1 Rational Closure and a Correspondence Result

Rational closure is a form of inferential closure based on modular entailment $\models_{\mathsf{mod}}$, but it extends its inferential power. Such an extension of modular entailment is obtained by formalising the already mentioned principle of *presumption of typicality* [69, Section 3.1]. That is, under possibly incomplete information, we always assume that we are dealing with the most typical possible situation that is compatible with the information at our disposal. We first define what it means for a concept to be *exceptional*, a notion that is central to the definition of rational closure:

Definition 23 (Exceptionality [[61]: Definition 10]). Let $\mathcal{KB}$ be a defeasible knowledge base and $C \in \mathcal{L}$. We say C is **exceptional** in $\mathcal{KB}$ if $\mathcal{KB} \models_{\mathsf{mod}} \top \sqsubsim \neg C$. A DCI $C \sqsubsim D$ is exceptional in $\mathcal{KB}$ if C is exceptional in $\mathcal{KB}$.

A concept C is considered exceptional in a knowledge base $\mathcal{KB}$ if it is not possible to have a modular model of $\mathcal{KB}$ in which there is a typical object (i.e., an object at least as typical as all the others) that is in the interpretation of C. This is expressed by requiring $\top \sqsubsim \neg C$ to be modularly entailed by $\mathcal{KB}$. Note that $\top \sqsubsim \neg C$ is not equivalent to $C \sqsubsim \bot$ in preferential interpretations. In fact, it follows easily from Lemma 2 that $C \sqsubsim \bot$ is equivalent to $C \sqsubseteq \bot$ in preferential interpretations.

Intuitively, a DCI is exceptional if it does not concern the most typical objects, i.e., it is about less normal (or exceptional) ones. This is an intuitive translation of the notion of exceptionality used by Lehmann and Magidor [70] in the propositional framework, and has already been used by Casini and Straccia [45] and Giordano et al. [60] in their investigations into defeasible reasoning for description logics.

Applying the notion of exceptionality iteratively, we associate with every concept C a *rank* in $\mathcal{KB}$, which we denote by $\mathsf{rank}_{\mathcal{KB}}(C)$. We extend this to DCIs and associate with every statement $C \sqsubsim D$ a rank, denoted $\mathsf{rank}_{\mathcal{KB}}(C \sqsubsim D)$:

(1) Let $\mathsf{rank}_{\mathcal{KB}}(C) = 0$, if C is not exceptional in $\mathcal{KB}$, and let $\mathsf{rank}_{\mathcal{KB}}(C \sqsubsim D) = 0$ for every DCI having C in the LHS, with $\mathsf{rank}_{\mathcal{KB}}(C) = 0$. The set of DCIs in $\mathcal{D}$ with rank 0 is denoted as $\mathcal{D}_0^{\mathsf{rank}}$.
(2) Let $\mathsf{rank}_{\mathcal{KB}}(C) = 1$, if C does not have a rank of 0 and it is not exceptional in the knowledge base $\mathcal{KB}^1$ composed of $\mathcal{T}$ and the exceptional part of $\mathcal{D}$, that is, $\mathcal{KB}^1 = \langle \mathcal{T}, \mathcal{D} \setminus \mathcal{D}_0^{\mathsf{rank}} \rangle$. If $\mathsf{rank}_{\mathcal{KB}}(C) = 1$, then let $\mathsf{rank}_{\mathcal{KB}}(C \sqsubsim D) = 1$ for every DCI $C \sqsubsim D$. The set of DCIs in $\mathcal{D}$ with rank 1 is denoted $\mathcal{D}_1^{\mathsf{rank}}$.
(3) In general, for $i > 0$, a concept C is assigned a rank of i if it does not have a rank of $i-1$ and it is not exceptional in $\mathcal{KB}^i = \langle \mathcal{T}, \mathcal{D} \setminus \bigcup_{j=0}^{i-1} \mathcal{D}_j^{\mathsf{rank}} \rangle$. If $\mathsf{rank}_{\mathcal{KB}}(C) = i$, then $\mathsf{rank}_{\mathcal{KB}}(C \sqsubsim D) = i$, for every DCI $C \sqsubsim D$ having C in the LHS. The set of DCIs in $\mathcal{D}$ with rank i is denoted $\mathcal{D}_i^{\mathsf{rank}}$.
(4) By iterating the previous steps, we eventually reach a (possibly empty) subset $\mathcal{E} \subseteq \mathcal{D}$ such that all the DCIs in $\mathcal{E}$ are exceptional (since $\mathcal{D}$ is finite, we must reach such a point). We define the rank of the DCIs in $\mathcal{E}$ (if there are any) as ∞, and the set $\mathcal{E}$ is denoted $\mathcal{D}_\infty^{\mathsf{rank}}$. Moreover, we set $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$ for every C in the LHS of some DCI in $\mathcal{D}_\infty^{\mathsf{rank}}$.

The notion of rank can also be extended to GCIs as follows: $\mathsf{rank}_{\mathcal{KB}}(C \sqsubseteq D) = \mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D)$.

Following on the procedure above, the defeasible TBox $\mathcal{D}$ is partitioned into a finite sequence $\langle \mathcal{D}_0^{\mathsf{rank}}, \ldots, \mathcal{D}_n^{\mathsf{rank}}, \mathcal{D}_\infty^{\mathsf{rank}} \rangle$ ($n \geq 0$), where $\mathcal{D}_\infty^{\mathsf{rank}}$ may possibly be empty. So, through this procedure we can assign a rank to every DCI.

We can check that for a concept C has a rank of ∞ iff it is not satisfiable in any modular model of $\mathcal{KB}$, that is, $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq \bot$.

Lemma 10. *For every knowledge base $\mathcal{KB}$ and every concept C, $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$ iff $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq \bot$.*

Example 3. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, where $\mathcal{T}$ and $\mathcal{D}$ are as in Example 1, i.e., $\mathcal{T} = \{\mathsf{EmpStud} \sqsubseteq \mathsf{Student}\}$ and

$$\mathcal{D} = \left\{ \begin{array}{c} \mathsf{Student} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \precsim \exists\mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax} \end{array} \right\}$$

Examining the concepts on the LHS of each DCI in $\mathcal{KB}$, one can verify that Student is not exceptional w.r.t. $\mathcal{KB}$. Therefore, $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Student}) = 0$. We also find that $\mathsf{rank}_{\mathcal{KB}}(\mathsf{EmpStud}) \neq 0$ and $\mathsf{rank}_{\mathcal{KB}}(\mathsf{EmpStud} \sqcap \mathsf{Parent}) \neq 0$, because both concepts are exceptional w.r.t. $\mathcal{KB}$. Hence, $\mathcal{D}_0^{\mathsf{rank}} = \{\mathsf{Student} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}\}$ and $\mathcal{KB}^0 = \mathcal{T} \cup \mathcal{D}_0^{\mathsf{rank}}$.

$\mathcal{KB}^1$ is composed of $\mathcal{T}$ and $\mathcal{D} \setminus \mathcal{D}_0^{\mathsf{rank}}$. We find that EmpStud is *not* exceptional w.r.t. $\mathcal{KB}^1$, and therefore $\mathsf{rank}_{\mathcal{KB}}(\mathsf{EmpStud}) = 1$. Since $\mathsf{EmpStud} \sqcap \mathsf{Parent}$ is exceptional w.r.t. $\mathcal{KB}^1$, $\mathsf{rank}_{\mathcal{KB}}(\mathsf{EmpStud} \sqcap \mathsf{Parent}) \neq 1$. Thus $\mathcal{D}_1^{\mathsf{rank}} = \{\mathsf{EmpStud} \precsim \exists\mathsf{pays}.\mathsf{Tax}\}$. Similarly, $\mathcal{KB}^2$ is composed of $\mathcal{T}$ and $\{\mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}\}$. We have that $\mathsf{EmpStud} \sqcap \mathsf{Parent}$ is not exceptional w.r.t. $\mathcal{KB}^2$, and therefore $\mathsf{rank}_{\mathcal{KB}}(\mathsf{EmpStud} \sqcap \mathsf{Parent}) = 2$. Finally, for this example, $\mathcal{D}_\infty^{\mathsf{rank}} = \emptyset$.

Adapting Lehmann and Magidor's construction for propositional logic [70], the rational closure of a defeasible knowledge base $\mathcal{KB}$ is defined as follows:

Definition 24 (Rational Closure). Let $\mathcal{KB}$ be a defeasible knowledge base and $C, D \in \mathcal{L}$.

(1) $C \precsim D$ is in the rational closure of $\mathcal{KB}$ if

$$\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) < \mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D) \text{ or } \mathsf{rank}_{\mathcal{KB}}(C) = \infty.$$

(2) $C \sqsubseteq D$ is in the rational closure of $\mathcal{KB}$ if $\mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D) = \infty$.

Informally, the definition above says that $C \precsim D$ is in the rational closure of $\mathcal{KB}$ if the modular models of $\mathcal{KB}$ tell us that some instances of $C \sqcap D$ are more plausible than all instances of $C \sqcap \neg D$, while $C \sqsubseteq D$ is in the rational closure of $\mathcal{KB}$ if the instances of $C \sqcap \neg D$ are impossible.

Example 3 (continued). Applying the definition above to the knowledge base in Example 3, we can verify that $\mathsf{Student} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}$ is in the rational closure of $\mathcal{KB}$, because $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Student} \sqcap \neg\exists\mathsf{pays}.\mathsf{Tax}) = 0$ and $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Student} \sqcap \exists\mathsf{pays}.\mathsf{Tax}) > 0$. The latter can be derived from the fact that $\mathsf{Student} \sqcap \exists\mathsf{pays}.\mathsf{Tax}$ is exceptional w.r.t. $\mathcal{KB}$. Similarly, one can derive that both DCIs $\mathsf{EmpStud} \precsim \exists\mathsf{pays}.\mathsf{Tax}$ and $\mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}$ are in the rational closure of $\mathcal{KB}$ as well.

We now state the main result of the present section, which provides an answer to the question raised at the end of Section 4.2. (The proof can be found in Appendix E.)

Theorem 4. *Let $\mathcal{KB}$ be a defeasible knowledge base having a modular model. A statement α is in the rational closure of $\mathcal{KB}$ iff $\mathcal{KB} \models_{\mathsf{rat}} \alpha$.*

An easy corollary of this result is that rational closure preserves the equivalence between GCIs of the form $C \sqsubseteq D$ and their defeasible counterparts ($C \sqcap \neg D \mathrel{\underset{\sim}{\sqsubset}} \bot$).

Corollary 3. *$C \sqsubseteq D$ is in the rational closure of a defeasible knowledge base $\mathcal{KB}$ iff $C \sqcap \neg D \mathrel{\underset{\sim}{\sqsubset}} \bot$ is in the restriction of the closure of $\mathcal{KB}$ under rational entailment to defeasible concept inclusions.*

Rational entailment from a knowledge base can therefore be formulated as membership checking of the rational closure of the knowledge base. Of course, from an application-oriented point of view, this raises the question of how to compute membership of the rational closure of a knowledge base, and what is the complexity thereof. This is precisely the topic of the next section.

5.2 Rational Entailment Checking

We now present an algorithm to effectively check the rational entailment of a DCI from a defeasible knowledge base. Our algorithm is a modification of the one given by Casini and Straccia [45] for defeasible $\mathcal{ALC}$. Their algorithm had to be modified in two ways. First, their computation of exceptionality had to be adapted (see below). Second, their algortihm does not always give back the correct result in case $\mathcal{D}_\infty^{\text{rank}} \neq \emptyset$ — cf. Item 4 in the description in Section 5.1.

Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be a defeasible knowledge base. The first step of the algorithm is to assign a rank to each DCI in $\mathcal{D}$. Central to this step is the exceptionality function Exceptional($\cdot$), which computes the semantic notion of exceptionality of Definition 23. The function makes use of the notion of *materialisation* to reduce concept exceptionality checking to entailment checking:

Definition 25 (Materialisation). Let $\mathcal{D}$ be a set of DCIs. With $\overline{\mathcal{D}} =_{\text{def}} \{\neg C \sqcup D \mid C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}\}$, we denote the **materialisation** of $\mathcal{D}$.

We can show that, given $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and $\mathcal{D}' \subseteq \mathcal{D}$, if $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}'} \sqsubseteq \neg C$, where $\models$ denotes *classical* $\mathcal{ALC}$ entailment, a DCI $C \mathrel{\underset{\sim}{\sqsubset}} D$ is exceptional w.r.t. $\mathcal{T} \cup \mathcal{D}'$, thereby justifying the use of Line 3 of function Exceptional($\cdot$). This is where our algorithm differs from that of Casini and Straccia [45]. Their check for exceptionality involved checking whether $\mathcal{T} \cup \{\top \sqsubseteq E \mid E \in \overline{\mathcal{D}'}\} \models \top \sqsubseteq \neg C$. The proof of the following lemma can be found in Appendix E.

Lemma 11. *For $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, if $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C$, then $C \mathrel{\underset{\sim}{\sqsubset}} D$ is exceptional w.r.t. $\mathcal{T} \cup \mathcal{D}$.*

Given a set of DCIs $\mathcal{D}' \subseteq \mathcal{D}$, Exceptional($\mathcal{T}, \mathcal{D}'$) returns a subset $\mathcal{E}$ of $\mathcal{D}'$ such that $\mathcal{E}$ is exceptional w.r.t. $\mathcal{T} \cup \mathcal{D}'$.

Function Exceptional($\mathcal{T}, \mathcal{D}'$)

Input: $\mathcal{T}$ and $\mathcal{D}' \subseteq \mathcal{D}$
Output: $\mathcal{E} \subseteq \mathcal{D}'$ such that $\mathcal{E}$ is exceptional w.r.t. $\mathcal{T} \cup \mathcal{D}'$
1 $\mathcal{E} := \emptyset$;
2 **foreach** $C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}'$ **do**
3   **if** $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}'} \sqsubseteq \neg C$ **then**
4         $\mathcal{E} := \mathcal{E} \cup \{C \mathrel{\underset{\sim}{\sqsubset}} D\}$
5 **return** $\mathcal{E}$

While the converse of Lemma 11 does not hold, it follows from Lemma 13 below that this reduction to classical entailment checking, when applied iteratively (lines 4–14 in Algorithm ComputeRanking($\cdot$)), fully captures the semantic notion of exceptionality of Definition 23.

Example 3 (continued). If we feed the knowledge base in Example 3 to the function Exceptional($\cdot$), then we obtain the following output:

$$\mathcal{E} = \{\text{EmpStud} \mathrel{\underset{\sim}{\sqsubset}} \exists\text{pays.Tax}, \text{EmpStud} \sqcap \text{Parent} \mathrel{\underset{\sim}{\sqsubset}} \neg\exists\text{pays.Tax}\}.$$

This is because both concepts on the LHS of the DCIs in $\mathcal{D}'$ are exceptional w.r.t. $\mathcal{KB}$ in Example 3.

We now describe the overall ranking algorithm, presented in the function ComputeRanking($\cdot$) below. The algorithm makes a finite sequence of calls to the function Exceptional($\cdot$), starting from the knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$. The algorithm terminates with a partitioning of the axioms in the DTBox, from which a ranking of axioms can easily be obtained.

Function ComputeRanking($\mathcal{KB}$)

Input: $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$
Output: $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ and an exceptionality ranking $\mathcal{E}$
1 $\mathcal{T}^* := \mathcal{T}$;
2 $\mathcal{D}^* := \mathcal{D}$;
3 **repeat**
4     $i := 0$;
5     $\mathcal{E}_0 := \mathcal{D}^*$;
6     $\mathcal{E}_1 :=$ Exceptional($\mathcal{T}^*, \mathcal{E}_0$);
7   **while** $\mathcal{E}_{i+1} \neq \mathcal{E}_i$ **do**
8         $i := i + 1$;
9         $\mathcal{E}_{i+1} :=$ Exceptional($\mathcal{T}^*, \mathcal{E}_i$);
10     $\mathcal{D}^*_\infty := \mathcal{E}_i$;
11     $\mathcal{T}^* := \mathcal{T}^* \cup \{C \sqsubseteq D \mid C \sqsubsim D \in \mathcal{D}^*_\infty\}$;
12     $\mathcal{D}^* := \mathcal{D}^* \setminus \mathcal{D}^*_\infty$;
13 **until** $\mathcal{D}^*_\infty = \emptyset$;
14 $\mathcal{E} := (\mathcal{E}_0, \ldots, \mathcal{E}_{i-1})$;
15 **return** ($\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*, \mathcal{E}$);

We initialise $\mathcal{T}^*$ to $\mathcal{T}$ and $\mathcal{D}^*$ to $\mathcal{D}$ (Lines 1 and 2 of ComputeRanking($\cdot$)). We then repeatedly invoke the function Exceptional($\cdot$) to obtain a sequence of sets of DCIs $\mathcal{E}_0, \mathcal{E}_1, \ldots$, where $\mathcal{E}_0 = \mathcal{D}^*$ and each $\mathcal{E}_{i+1}$ is the set of exceptional axioms in $\mathcal{E}_i$ (Lines 4–14 of ComputeRanking($\cdot$)).

Now, let $C_{\mathcal{D}^*} =_{\text{def}} \{C \mid C \sqsubsim D \in \mathcal{D}^*\}$, i.e., $C_{\mathcal{D}^*}$ is the set of all *antecedents* of DCIs in $\mathcal{D}^*$. The exceptionality ranking of the DCIs in $\mathcal{D}^*$ computed by Exceptional($\cdot$) makes use of $\mathcal{T}^*$, $\overline{\mathcal{D}^*}$, and $C_{\mathcal{D}^*}$. That is, it checks, for each concept $C \in C_{\mathcal{D}^*}$, whether $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{D}^*} \sqsubseteq \neg C$. In case C is exceptional, every DCI $C \sqsubsim D \in \mathcal{D}^*$ is exceptional w.r.t. $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ and is added to the set $\mathcal{E}_1$.

If $\mathcal{E}_1 \neq \mathcal{E}_0$, then we call Exceptional($\cdot$) for $\mathcal{T}^* \cup \mathcal{E}_1$, defining the set $\mathcal{E}_2$, and so on. Hence, given $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$, we construct a sequence $\mathcal{E}_0, \mathcal{E}_1, \ldots$ in the following way:

- $\mathcal{E}_0 := \mathcal{D}^*$;
- $\mathcal{E}_{i+1} :=$ Exceptional($\mathcal{T}^*, \mathcal{E}_i$), for $i \geq 0$.

Example 3 (continued). Using the knowledge base of Example 3, we initialise $\mathcal{T}^* = \{\mathsf{EmpStud} \sqsubseteq \mathsf{Student}\}$ and

$$\mathcal{D}^* = \left\{ \begin{array}{c} \mathsf{Student} \sqsubsim \neg\exists\mathsf{pays.Tax}, \\ \mathsf{EmpStud} \sqsubsim \exists\mathsf{pays.Tax}, \\ \mathsf{EmpStud} \sqcap \mathsf{Parent} \sqsubsim \neg\exists\mathsf{pays.Tax} \end{array} \right\}.$$

We then obtain the following exceptionality sequence:

$$\mathcal{E}_0 = \left\{ \begin{array}{c} \mathsf{Student} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \precsim \exists\mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax} \end{array} \right\},$$

$$\mathcal{E}_1 = \left\{ \begin{array}{c} \mathsf{EmpStud} \precsim \exists\mathsf{pays}.\mathsf{Tax}, \\ \mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax} \end{array} \right\},$$

$$\mathcal{E}_2 = \{\mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}\}.$$

Since $\mathcal{D}^*$ is finite, the construction will eventually terminate with a fixed point $\mathcal{E}_{\mathsf{fix}}$ corresponding to Exceptional$(\mathcal{T}^*, \mathcal{E}_{\mathsf{fix}})$. If this fixed point is non-empty, then the axioms in there are said to have infinite rank. We therefore set $\mathcal{D}^*_\infty =_{\mathrm{def}} \mathcal{E}_{\mathsf{fix}}$ (Line 11 of ComputeRanking$(\cdot)$), and the classical translations of these axioms are moved to the TBox. Hence, we redefine the knowledge base in the following way (Lines 12 and 13 of ComputeRanking$(\cdot)$):

- $\mathcal{T}^* := \mathcal{T}^* \cup \{C \sqsubseteq D \mid C \precsim D \in \mathcal{D}^*_\infty\}$;
- $\mathcal{D}^* := \mathcal{D}^* \setminus \mathcal{D}^*_\infty$.

Function ComputeRanking$(\cdot)$ must terminate, since $\mathcal{D}$ is finite, and at every iteration, $\mathcal{D}^*$ becomes smaller (hence, we have at most $|\mathcal{D}|$ iterations). In the end, we obtain a knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$, which is modularly equivalent to the original knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ (see Lemma 12 below), in which $\mathcal{D}^*$ has no DCIs of infinite rank (all the classical knowledge implicit in the DTBox has been moved to the TBox). We say that such a knowledge base is in *rank normal form*.

We also obtain a final exceptionality sequence $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{i-1})$ (see Line 15 of the function ComputeRanking$(\cdot)$). Given $\mathcal{E}$, it is possible to partition the set $\mathcal{D}^*$ into the sets $\mathcal{D}_0, \ldots, \mathcal{D}_n$, for some $n = i - 1 \geq 0$:

- For every j, $0 \leq j \leq n$, $\mathcal{D}_j := \mathcal{E}_j \setminus \mathcal{E}_{j+1}$;
- $R := (\mathcal{D}_0, \ldots, \mathcal{D}_n)$.

The sequence R is a partition of the DTBox according to the level of exceptionality of each defeasible inclusion in it.

Example 3 (continued). For $\mathcal{KB}$ as in Example 3, we obtain the partition $R = \{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$, where $\mathcal{D}_0 = \{\mathsf{Student} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}\}$, $\mathcal{D}_1 = \{\mathsf{EmpStud} \precsim \exists\mathsf{pays}.\mathsf{Tax}\}$ and $\mathcal{D}_2 = \{\mathsf{EmpStud} \sqcap \mathsf{Parent} \precsim \neg\exists\mathsf{pays}.\mathsf{Tax}\}$.

At this stage, we have moved all the classical information implicit the DTBox to the TBox, and ranked all the remaining DCIs, where the rank of a DCI is the index of the unique partition to which it belongs, defined as follows:

Definition 26 (Ranking). For every $C, D \in \mathcal{L}$:

- $\mathsf{rk}(C) =_{\mathrm{def}} i$, $0 \leq i \leq n$, if $\mathcal{E}_i$ is the first element in the sequence $(\mathcal{E}_0, \ldots, \mathcal{E}_n)$ s.t. $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \bot$ does *not* hold;
- $\mathsf{rk}(C) =_{\mathrm{def}} \infty$, if there is no such $\mathcal{E}_i$;
- $\mathsf{rk}(C \precsim D) =_{\mathrm{def}} \mathsf{rk}(C)$.

Remark 1. For every $i \leq j \leq n$, $\models \bigsqcap \overline{\mathcal{E}_j} \sqsubseteq \bigsqcap \overline{\mathcal{E}_i}$.

Remark 2. For every $i < j \leq n$, $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$.

To summarise, we transform our initial knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, obtaining a modularly equivalent knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ (see Lemma 12 below) and a ranking of DCIs in the form of a partitioning of $\mathcal{D}^*$. The main difference between ComputeRanking$(\cdot)$ and the analogous procedure by Casini and Straccia [45] is the reiteration of the ranking procedure until $\mathcal{D}^*_\infty = \emptyset$ (lines 4–14 in ComputeRanking$(\cdot)$). While the two procedures behave identically in the case where there are no DCIs $C \sqsubsetsim D$ s.t. $\mathrm{rank}_{\mathcal{KB}}(C \sqsubsetsim D) = \infty$ in $\mathcal{D}$, the procedure by Casini and Straccia [45] did not handle all the cases correctly in which there is classical information implicit in the DTBox. The example below shows the difference between the two procedures.

Example 4 ([49], Example 4). Let $\mathcal{KB} = \langle \mathcal{T}, \mathcal{D} \rangle$ be an ontology with

$$\begin{aligned}
\mathcal{T} &= \{\ A \sqsubseteq B, \\
&\qquad B \sqcap D \sqsubseteq \bot\ \}, \\
\mathcal{D} &= \{\ B \sqsubsetsim C, \\
&\qquad A \sqsubsetsim D, \\
&\qquad E \sqsubsetsim \exists r.A\ \}.
\end{aligned}$$

It can be verified that the execution of `ComputeRanking`$(\mathcal{KB})$ is as follows:

$$\begin{aligned}
& \mathcal{T}^* = \mathcal{T}, \mathcal{D}^* = \mathcal{D}, \\
\texttt{First iteration}: i = 0 \quad & \mathcal{E}_0 = \mathcal{D}^*, \mathcal{E}_1 = \{A \sqsubsetsim D\} \\
i = 1 \quad & \mathcal{E}_2 = \{A \sqsubsetsim D\} \text{ (end while)} \\
& \mathcal{D}_\infty = \mathcal{E}_2 = \{A \sqsubsetsim D\} \\
& \mathcal{D}^* = \mathcal{D}^* \setminus \{A \sqsubsetsim D\} = \{B \sqsubsetsim C, E \sqsubsetsim \exists r.A\} \\
& \mathcal{T}^* = \mathcal{T}^* \cup \{A \sqsubseteq D\} = \{A \sqsubseteq B, A \sqsubseteq D, B \sqcap D \sqsubseteq \bot\}, \\
\\
\texttt{Second iteration}: i = 0 \quad & \mathcal{E}_0 = \mathcal{D}^*, \mathcal{E}_1 = \{E \sqsubsetsim \exists r.A\} \\
i = 1 \quad & \mathcal{E}_2 = \{E \sqsubsetsim \exists r.A\ \} \text{ (end while)} \\
& \mathcal{D}_\infty = \mathcal{E}_2 = \{E \sqsubsetsim \exists r.A\} \\
& \mathcal{D}^* = \mathcal{D}^* \setminus \{E \sqsubsetsim \exists r.A\ \} = \{B \sqsubsetsim C\} \\
& \mathcal{T}^* = \mathcal{T}^* \cup \{E \sqsubseteq \exists r.A\} = \{A \sqsubseteq B, A \sqsubseteq D, B \sqcap D \sqsubseteq \bot, E \sqsubseteq \exists r.A\}, \\
\\
\texttt{Third iteration}: i = 0 \quad & \mathcal{E}_0 = \mathcal{D}^*, \mathcal{E}_1 = \emptyset \\
i = 1 \quad & \mathcal{E}_2 = \emptyset \text{ (end while)} \\
& \mathcal{D}_\infty = \mathcal{E}_2 \\
& \mathcal{D}^* = \mathcal{D}^* \setminus \emptyset = \{B \sqsubsetsim C\} \\
& \mathcal{T}^* = \mathcal{T}^* \cup \emptyset = \{E \sqsubseteq \exists r.A\} = \{A \sqsubseteq B, A \sqsubseteq D, B \sqcap D \sqsubseteq \bot, E \sqsubseteq \exists r.A\} \\
& \text{(end repeat)}, \\
\\
\texttt{for}\ j = 1 \quad & \mathcal{D}_0 = \mathcal{E}_0 \setminus \mathcal{E}_1 = \{B \sqsubsetsim C\}.
\end{aligned}$$

Therefore, `ComputeRanking`($\mathcal{KB}$) terminates with

$$\begin{aligned}\mathcal{T}^* &= \{\ A \sqsubseteq B, A \sqsubseteq D, B \sqcap D \sqsubseteq \bot, E \sqsubseteq \exists r.A\ \},\\ \mathcal{D}^* &= \{\ B \mathrel{\sqsubset\!\!\!\sim} C\ \},\\ \mathcal{D}_0 &= \{\ B \mathrel{\sqsubset\!\!\!\sim} C\ \}.\end{aligned}$$

The only defeasible axiom in $\mathcal{D}^*$ is $B \mathrel{\sqsubset\!\!\!\sim} C$, which has rank 0. Axioms $A \mathrel{\sqsubset\!\!\!\sim} D$ and $E \mathrel{\sqsubset\!\!\!\sim} \exists r.A$ have rank ∞ instead, and so are substituted by the classical counterparts $A \sqsubseteq D$ and $E \sqsubseteq \exists r.A$; we need to iterate the loop in lines 5–13 in procedure `ComputeRanking` more than once to determine such ranking values.

The reader can notice in the example above that the iteration of the exceptionality procedure in the lines 5–13 in the procedure ComputeRanking($\cdot$) was necessary because of the presence of the role r in the axiom $E \mathrel{\sqsubset\!\!\!\sim} \exists r.A$; if the axiom had the form $E \mathrel{\sqsubset\!\!\!\sim} A$, it would have been ranked in $\mathcal{D}_\infty$ at the first iteration. The decision procedure originally presented by Casini and Straccia ([45, p.83], Steps 3 and 4) lacks the iteration of the exceptionality procedure in lines 5–13: consequently it is not able to manage this kind of situations (that is, the axiom $E \sqsubseteq \exists r.A$ and the concept E would have been considered of rank 0), while in all the other cases it behaves exactly like the procedure ComputeRanking($\cdot$) presented here.

Our present procedure corresponds to the semantic constructions we have introduced above. in particular, Lemma 45 in Appendix E and Lemma 13 below prove that the procedure here is correct w.r.t. the semantics.

Given the knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$, we can now define the main algorithm for deciding whether a DCI $C \mathrel{\sqsubset\!\!\!\sim} D$ is in the rational closure of $\mathcal{KB}$. To do that, we use the same approach as in the function Exceptional($\cdot$), that is, given $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ and our sequence of sets $\mathcal{E}_0, \dots, \mathcal{E}_n$, we use the TBox $\mathcal{T}^*$ and the sets of conjunctions of materialisations $\bigsqcap \overline{\mathcal{E}_0}, \dots, \bigsqcap \overline{\mathcal{E}_n}$.

Definition 27 (Rational Deduction). Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and let $C, D \in \mathcal{L}$. We say that $C \mathrel{\sqsubset\!\!\!\sim} D$ is **rationally deducible** from $\mathcal{KB}$, denoted $\mathcal{KB} \vdash_{\text{rat}} C \mathrel{\sqsubset\!\!\!\sim} D$, if $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$, where $\bigsqcap \overline{\mathcal{E}_i}$ is the first element of the sequence $\bigsqcap \overline{\mathcal{E}_0}, \dots, \bigsqcap \overline{\mathcal{E}_n}$ s.t. $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg C$ does not hold. If there is no such element, then $\mathcal{KB} \vdash_{\text{rat}} C \mathrel{\sqsubset\!\!\!\sim} D$ if $\mathcal{T}^* \models C \sqsubseteq D$.

Observe that $\mathcal{KB} \vdash_{\text{rat}} C \sqsubseteq D$ if and only if $\mathcal{KB} \vdash_{\text{rat}} C \sqcap \neg D \mathrel{\sqsubset\!\!\!\sim} \bot$, i.e., if and only if $\mathcal{KB} \vdash_{\text{rat}} C \sqcap \neg D \sqsubseteq \bot$ (that is to say, $\mathcal{T}^* \models C \sqsubseteq D$).

The algorithm corresponding to the steps above is presented in the function RationalClosure($\cdot$) below.

Function RationalClosure($\mathcal{KB}$, α)

Input: $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and a query $\alpha = C \mathrel{\sqsubset\!\!\!\sim} D$.
Output: `true` if $\mathcal{KB} \vdash_{\text{rat}} C \mathrel{\sqsubset\!\!\!\sim} D$, `false` otherwise

($\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*, \mathcal{E} = (\mathcal{E}_0, \dots, \mathcal{E}_n)$) := ComputeRanking($\mathcal{KB}$);
$i := 0$;
while $\mathcal{T}^* \models \sqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \bot$ *and* $i \leq n$ **do**
  $i := i + 1$;
if $i \leq n$ **then**
  return $\mathcal{T}^* \models \sqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$;
else
  return $\mathcal{T}^* \models C \sqsubseteq D$;

Example 4 (continued). Let $\mathcal{KB}$ be as in Example 3 and assume we want to check whether the DCI $\mathsf{EmpStud} \precsim \exists\mathsf{pays.Tax}$ is in the rational closure of $\mathcal{KB}$. Then, the while-loop on Line 2 of function RationalClosure$(\cdot)$ terminates when $i = 1$. At this stage, $\bigsqcap \overline{\mathcal{E}_i} = (\neg\mathsf{EmpStud} \sqcup \exists\mathsf{pays.Tax}) \sqcap (\neg\mathsf{EmpStud} \sqcup \neg\mathsf{Parent} \sqcup \neg\exists\mathsf{pays.Tax})$. Given this, one can check that $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \bot$ does not hold, i.e., it is not the case that $\{\mathsf{EmpStud} \sqsubseteq \mathsf{Student}\} \models (\neg\mathsf{EmpStud} \sqcup \exists\mathsf{pays.Tax}) \sqcap (\neg\mathsf{EmpStud} \sqcup \neg\mathsf{Parent} \sqcup \neg\exists\mathsf{pays.Tax}) \sqcap \mathsf{EmpStud} \sqsubseteq \bot$. Finally, it is easy to confirm that we do not have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$, i.e., it is not the case that $\{\mathsf{EmpStud} \sqsubseteq \mathsf{Student}\} \models (\neg\mathsf{EmpStud} \sqcup \exists\mathsf{pays.Tax}) \sqcap (\neg\mathsf{EmpStud} \sqcup \neg\mathsf{Parent} \sqcup \neg\exists\mathsf{pays.Tax}) \sqcap \mathsf{EmpStud} \sqsubseteq \exists\mathsf{pays.Tax}$.

Before we state the main theorem of this section, we need to establish the correspondence between the ranking function $\mathsf{rank}_{\mathcal{KB}}(\cdot)$ presented in Section 5.1 in the construction of the rational closure of $\mathcal{KB}$ and linked by Theorem 4 to the definition of rational entailment, and the ranking function $\mathsf{rk}(\cdot)$ of Definition 26 used in the above algorithm. We also need to establish that the normalisation of a knowledge base by our algorithm maintains modular equivalence. The proofs of the following lemmas, as well as a number of prerequisite results, are in Appendix E.

Lemma 12. *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and let $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ be obtained from $\mathcal{KB}$ through function* ComputeRanking$(\cdot)$. *Then $\mathcal{KB}$ and $\mathcal{KB}^*$ are modularly equivalent.*

Lemma 13. *For every defeasible knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and every $C \in \mathcal{L}$,* $\mathsf{rank}_{\mathcal{KB}}(C) = \mathsf{rk}(C)$.

Now, we can state the main theorem, which links rational entailment to rational deduction via Theorem 4. (The proof can be found in Appendix E.)

Theorem 5. *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and let $C, D \in \mathcal{L}$. Then $\mathcal{KB} \vdash_{\mathsf{rat}} C \precsim D$ iff $\mathcal{KB} \models_{\mathsf{rat}} C \precsim D$.*

As an immediate consequence, we have that the function RationalClosure$(\cdot)$ is correct w.r.t. the definition of rational closure in Definition 24.

Corollary 4. *Checking rational entailment is* EXPTIME*-complete.*

Hence, entailment checking for defeasible ontologies is just as hard as classical subsumption checking.

We conclude this section by noting that although rational closure is viewed as an appropriate form of defeasible reasoning, it does have its limitations, the first of which is that it does not satisfy the *presumption of independence* [69, Section 3.1]. To consider a well-worn example, suppose we know that birds usually fly and usually have wings, that both penguins and robins are birds, and that penguins usually do not fly. That is, we have the following knowledge base: $\mathcal{KB} = \{\mathsf{Bird} \precsim \mathsf{Flies}, \mathsf{Bird} \precsim \mathsf{HasWings}, \mathsf{Penguin} \sqsubseteq \mathsf{Bird}, \mathsf{Robin} \sqsubseteq \mathsf{Bird}, \mathsf{Penguin} \precsim \neg\mathsf{Flies}\}$. Rational closure allows us to conclude that robins usually have wings, since they are viewed as typical birds, thereby satisfying the presumption of typicality. But with penguins being atypical birds, rational closure does not allow us to conclude that penguins usually have wings, thus violating the presumption of independence, which, in this context, would require the atypicality of penguins w.r.t. flying to be independent of the typicality of penguins w.r.t. having wings.

This deficiency is well-known, and there are other forms of defeasible reasoning that can overcome this, most notably lexicographic closure [47], relevance closure [43], and inheritance-based closure [46]. But note that the presumption of independence is *propositional* in nature. In fact, the DL version of lexicographic closure is essentially a lifting to the DL case of a propositional solution to the problem [69].

What is perhaps of more interest is the inability of rational closure to deal with defeasibility relating to the *non-propositional* aspects of descriptions logics. For example, Pensel and Turhan [74, 75] have shown that rational closure across role expressions does not always support defeasible inheritance appropriately. Suppose we know that bosses are workers, do not have workers as their superiors, and are usually responsible. Furthermore, suppose we know that workers usually have bosses as their superiors. We thus have the knowledge base $\mathcal{KB} = \{\text{Boss} \sqsubseteq \text{Worker}, \text{Boss} \sqsubseteq \neg\exists\text{hasSuperior}.\text{Worker}, \text{Boss} \precsim \text{Responsible}, \text{Worker} \precsim \exists\text{hasSuperior}.\text{Boss}\}$. Since workers usually have bosses as their superiors, and bosses are usually responsible, one would expect to be able to conclude that workers usually have responsible superiors. But rational closure is unable to do so. From the perspective of the algorithm for rational closure, this can be traced back to the use of materialisation (Definition 25) when computing exceptionality, as Pensel and Turhan show. A more detailed semantic explanation for this inability is still forthcoming, though.

6 RELATED WORK

In a sense, the first investigations on non-monotonic reasoning in DL-based systems date back to the work by Brewka [22] and Cadoli et al. [40]. Other early proposals to introduce default-style rules into description logics include the work by Baader and Hollunder [2, 3], Padgham and Zhang [73], and Straccia [86], which are essentially based on Reiter's default logic [79].

Quantz and Royer [77] were probably the first to consider lifting the preferential approach to a DL setting. They propose a general framework for Preferential Default Description Logics (PDDL) based on an $\mathcal{ALC}$-like language by introducing a version of default subsumption and proposing a preferential semantics for it. Their semantics is based on a simplified version of standard DL interpretations. They assume all domains to be finite, which means their framework is much more restrictive than ours in this aspect. They also allow for the use of object names (something we do not do), and assume that the unique-name assumption holds for object names.

They focus on a version of entailment that they refer to as preferential entailment but that is to be distinguished from the version of preferential entailment that we have presented in this article. In what follows, we shall refer to their version as *QR-preferential entailment.*

QR-preferential entailment is concerned with what ought to follow from a set of DL statements, together with a set of default subsumption statements, and is parametrised by a fixed partial order on (simplified) DL interpretations. (That is, the ordering is on the set of DL interpretations, not on the elements of their respective domains.) They prove that any QR-preferential entailment satisfies the properties of a preferential consequence relation and, with some restrictions on the partial order, satisfies rational monotonicity as well. QR-preferential entailment can therefore be viewed as something in between the notions of preferential entailment (or modular entailment) and rational entailment. It is also worth noting that although QR-preferential entailment satisfies the properties of a preferential consequence relation, Quantz and Royer do not prove that QR-preferential entailment provides a characterisation of preferential consequence in the spirit of the representation results we have shown here.

Closely related to our work is that of Giordano et al. [57] who use preferential orderings on $\Delta^{\mathcal{I}}$ to define a typicality operator $\mathbf{T}(\cdot)$ on $\mathcal{ALC}$ concepts such that the expression $\mathbf{T}(C) \sqsubseteq D$ corresponds to our $C \precsim D$. They provide a version of a representation result for preferential orderings in terms of properties on selection functions (functions on the power set of the domain of interpretations), but not a representation result along the lines of those we have shown here. In the same work, the authors define a tableaux calculus for computing preferential entailment that relies on KLM-style rules. Their approach is then extended with a circumscriptive-like solution (see below), since it relies on the specification of a set of concepts for which atypical instances must be minimised [59].

Giordano and colleagues [60, 61] also consider an approach that is similar, but incomparable in terms of inferential power, to the aforementioned work by considering modular orderings on $\Delta^{\mathcal{I}}$ (i.e., modular interpretations) and then employing a version of a minimal-model semantics, in which some modular interpretations are preferred over others. This is similar in intuition to rational entailment, and in fact it has been proved that our *Big Ranked Model* (Definition 21) corresponds to the main semantic construction they propose [49, Proposition 30].

Regarding an algorithm for computing rational closure, Giordano et al. [62] consider a different approach, with an encoding that reduces defeasible subsumption (in their phrasing of this notion) to classical subsumption in $\mathcal{ALC}$. This encoding is the same as the one they use for reducing defeasible subsumption in $\mathcal{SHIQ}$ to classical subsumption in $\mathcal{SHIQ}$ [60]. More recently, they reported on a Protégé plugin based on this encoding [63]. Recently Bonatti has also investigated how to extend the rational closure construction to DLs that do not satisfy the *disjoint union model property* [12].

Outside the family of preferential systems, there are mature proposals based on circumscription for DLs [9, 10, 84]. The main drawback of these approaches is the burden on the ontology engineer to make appropriate decisions related to the (circumscriptive) fixing and varying of concepts and the priority of defeasible subsumption statements. Such choices can have a major effect on the conclusions drawn from the system, and can easily lead to counter-intuitive inferences. Moreover, the use of circumscription usually implies a considerable increase in computational complexity w.r.t. the underlying monotonic entailment relation. The comparison between the present work and proposals outside the preferential family is more an issue about the pros and cons of the different kinds of non-monotonic reasoning, rather than about their DL re-formulation. As stated in the introduction, the preferential approach has a series of desirable qualities that, to our knowledge, no other approach to non-monotonic reasoning shares.

A more recent proposal is the approach by Bonatti et al. [8, 11], which introduces a *normality* operator $\mathbf{N}(\cdot)$ on concepts. The resulting system, DL^{N}, is not based on the preferential approach, though, and as a consequence their closure operation does not allow defeasible subsumption to satisfy the preferential properties, but it satisfies some interesting properties on the meta-level. It also has the advantage of being computationally tractable for any tractable classical DL.

Lukasiewicz [71] proposes probabilistic versions of the description logics $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$. As a special case of these logics, he obtains a version of a logic with defeasible subsumption with a semantics based on that of the propositional version of lexicographic closure [69].

Casini and Straccia [45] define a decision procedure for $\mathcal{ALC}$ that is a version of an algorithm for computing rational closure for the propositional case, but simply lifted to the case for description logics. As such, their proposal has a syntactic characterisation, but lacks an appropriate semantics, a deficiency that the present article comes to remedy. The semantic approach presented here can be extended also to other forms of entailment proposed by them [46–48]. The main constructions and results in the present article have already been published in a technical report [23], and here we present them in a more coherent and insightful framework. In particular, the procedures presented here have been exploited by the authors in other publications. For example, recently Casini, Straccia and Meyer have used it also to characterise a decision procedure for defeasible $\mathcal{EL}_{\perp}$ [49], and some initial experimental results have already been published [44]. An application of our algorithms in the legal framework is under development at present [80].

Britz and Varzinczak [32, 36, 37] explore the notion of *defeasible modalities*, with which defeasible effects of actions, defeasible knowledge, obligations and others can be formalised and given an intuitive preferential semantics. The approach followed there only considers preferential entailment, but the semantic constructions are similar. This was extended [25] to a notion of *defeasible* *role restrictions* in a DL setting. The idea comprises extending the language of $\mathcal{ALC}$ with an additional construct $\forall\!\!\!\sim$. The semantics of a concept $\forall\!\!\!\sim r.C =_{\text{def}} \{x \in \Delta^{\mathcal{P}} \mid \min_{\prec^{\mathcal{P}}} r^{\mathcal{P}}(x) \subseteq C^{\mathcal{P}}\}$ is then given by all objects of $\Delta^{\mathcal{P}}$ such that all of their *minimal* r-related objects are C-instances. This is useful in situations where certain classical concept descriptions may be too strong.

Recently, Britz and Varzinczak have lifted the preferential semantics to also allow for orderings on role-interpretations [33, 35] that, in turn, induce multi-orderings on objects of the domain [34, 38, 39]. The latter give us the handle needed to introduce a notion of *context* in defeasible subsumption relations making typicality a relativised construct. The former provides a semantics for defeasible role inclusions of the form $r \sqsubseteq\!\!\!\sim s$ and for defeasible role assertions such as "r is usually transitive", "r and s are usually disjoint", as well as others.

Another recent proposal that uses contextual reasoning to cope with defeasibility has been developed by Bozzato and others [19, 20], extending with an overriding mechanism the *Contextualized Knowledge Repository (CKR) framework* [21].

Finally, there is the recent work of Pensel and Turhan [74, 75] mentioned in Section 5.2, the aim of which is to extend both rational closure and relevant closure with defeasible inheritance across role expressions in the description logic $\mathcal{EL}_{\perp}$. With their work being restricted to $\mathcal{EL}_{\perp}$, the semantics they propose is based on a form of canonical model similar to those frequently used for the $\mathcal{EL}$ family of DLs, and is therefore quite different from ours. A detailed comparison of their semantics with the one we provide in this article is left as future work.

7 CONCLUDING REMARKS

The main contributions of the work reported in the present article can be summarised as follows:

(1) The analysis of a simple and intuitive semantics for defeasible subsumption in description logics that is general enough to constitute the core framework within which to investigate non-monotonic extensions of DLs;
(2) A characterisation of preferential and rational subsumption relations, with the respective representation results, evidencing the fact that our semantic constructions are appropriate;
(3) An investigation of what an appropriate notion of entailment in a defeasible DL context means and the analysis of a suitable candidate, namely, rational entailment, and
(4) The formal connection between rational entailment, the notion of rational closure and an algorithm for its computation.

With regard to point (4) above, the main advantages of our approach are as follows: (*i*) it relies completely on classical entailment, i.e., entailment checking over defeasible ontologies can be reduced to a number of classical entailment checks over a rewritten ontology; (*ii*) it has computational complexity that is no worse than that of entailment checking in the classical underlying DL; and (*iii*) it is easily implementable, e.g., as a Protégé plugin,[3] of which the performance has been shown to scale well in practice [44]. In a companion paper [24], the framework described here is extended to include ABox reasoning, with more extensive experimental results confirming the initial promising results on scalability.

In Section 3.2, we briefly presented three properties involving quantifiers that are satisfied by preferential subsumption relations. For future work, we are interested in a more detailed study of properties involving quantifiers that are not satisfied by preferential subsumption relations (or not

[3] https://github.com/kodymoodley/defeasibleinferenceplatform.

satisfied by rational subsumption relations), such as (∀Or):

$$(\forall\text{Or})\ \frac{\forall r.C \precsim E,\ \forall r.D \precsim E}{\forall r.(C \sqcup D) \precsim E}.$$

Further topics for future research include the full integration of notions such as *typicality* for both concepts and roles [13–16, 57, 59, 87] and role-based defeasible constructors [33, 35, 38, 39] with the work presented here. The semantic constructions are compatible, but the rational entailment checking procedure of Section 5.2 would need to be generalised to deal with the additional defeasible constructs. Another avenue for future exploration is the study of belief revision for DLs via our results for rationality, somewhat mimicking the well-known connection between belief revision and rational consequence in the propositional case [41, 42, 53], thereby pushing the frontiers of theory change in logics that are more expressive than the propositional one.

APPENDICES

A PROOFS OF LEMMAS IN SECTION 3.2

NB: The results marked (∗) are introduced here in the Appendix, while they are omitted in the main text.

Lemma 2. *For every preferential interpretation $\mathcal{P}$, and every $C, D \in \mathcal{L}$, $\mathcal{P} \Vdash C \sqsubseteq D$ if and only if $\mathcal{P} \Vdash C \sqcap \neg D \precsim \bot$.*

Proof. From left to right, $\mathcal{P} \Vdash C \sqsubseteq D$ implies, by Lemma 1, that $(C \sqcap \neg D)^{\mathcal{P}} = \emptyset$. The latter implies that, for every concept E, $\mathcal{P} \Vdash C \sqcap \neg D \precsim E$, and, as a particular case, $\mathcal{P} \Vdash C \sqcap \neg D \precsim \bot$. From right to left, if it is not the case that $\mathcal{P} \Vdash C \sqsubseteq D$, then $(C \sqcap \neg D)^{\mathcal{P}} \neq \emptyset$. Let x be an object in $\min_{\prec^{\mathcal{P}}}(C \sqcap \neg D)^{\mathcal{P}}$: for $\mathcal{P} \Vdash C \sqcap \neg D \precsim \bot$, we should have also $x \in \bot^{\mathcal{P}}$, which is a contradiction. □

Definition 28 (Disjoint-Union Preferential Interpretation). Let S be a countable set and let $\mathscr{P} = \{\mathcal{P}_s = \langle \Delta^{\mathcal{P}_s}, \cdot^{\mathcal{P}_s}, \prec^{\mathcal{P}_s} \rangle \mid s \in S\}$ be a collection of preferential interpretations. The **disjoint union of** $\mathscr{P}$ is a tuple $\mathcal{U} =_{\text{def}} \langle \Delta^{\mathcal{U}}, \cdot^{\mathcal{U}}, \prec^{\mathcal{U}} \rangle$ where:

- $\Delta^{\mathcal{U}} =_{\text{def}} \{(x, s) \mid x \in \Delta^{\mathcal{P}_s} \text{ and } s \in S\}$;
- $A^{\mathcal{U}} =_{\text{def}} \{(x, s) \mid x \in A^{\mathcal{P}_s} \text{ and } s \in S\}$, for every $A \in \mathsf{C}$;
- $r^{\mathcal{U}} =_{\text{def}} \{((x, s), (y, s)) \mid (x, y) \in r^{\mathcal{P}_s} \text{ and } s \in S\}$, for every $r \in \mathsf{R}$;
- $\prec^{\mathcal{U}} =_{\text{def}} \{((x, s), (y, s)) \mid (x, y) \in \prec^{\mathcal{P}_s} \text{ and } s \in S\}$.

Lemma 14 (∗). *Let S and $\mathscr{P}$ be as in Definition 28 and let $\mathcal{U}$ be the latter's disjoint union. For every $C \in \mathcal{L}$, every $s \in S$, and every $x \in \Delta^{\mathcal{P}_s}$, $x \in C^{\mathcal{P}_s}$ if and only if $(x, s) \in C^{\mathcal{U}}$.*

Proof. For every $s \in S$, define $\mathsf{E}_s =_{\text{def}} \{(x, (x, s)) \mid x \in \Delta^{\mathcal{P}_s}\}$. We can easily show that E_s is a preferential bisimulation [37] between $\mathcal{P}_s$ and $\mathcal{U}$. The lemma is then proved by induction on the structure of concepts in the usual way [4]. □

It is easy to see that the following result also holds:

Lemma 15 (∗). *Let S and $\mathscr{P}$ be as in Definition 28 and let $\mathcal{U}$ be the latter's disjoint union. For every $C \in \mathcal{L}$, every $s \in S$, and every $x \in \Delta^{\mathcal{P}_s}$, $x \in \min_{\prec^{\mathcal{P}_s}} C^{\mathcal{P}_s}$ if and only if $(x, s) \in \min_{\prec^{\mathcal{U}}} C^{\mathcal{U}}$.*

Lemma 3. *Preferential interpretations are closed under disjoint union.*

Proof. Let $\mathcal{KB}$ be a defeasible knowledge base, let S and $\mathscr{P}$ be as in Definition 28 and such that $\mathcal{P}_s \Vdash \mathcal{KB}$, for every $\mathcal{P}_s \in \mathscr{P}$, and let $\mathcal{U}$ be the disjoint union of the models in $\mathscr{P}$. We have to show that $\mathcal{U} \Vdash \mathcal{KB}$. Assume that we do not have $\mathcal{U} \Vdash \mathcal{KB}$. Then there must be a DCI $C \precsim D \in \mathcal{KB}$ (recall Lemma 2) and an object $(x, s) \in \Delta^{\mathcal{U}}$ such that $(x, s) \in \min_{\prec^{\mathcal{U}}} C^{\mathcal{U}}$ but $(x, s) \notin D^{\mathcal{U}}$. From

Lemmas 14 and 15 above, it follows that $x \in \min_{\prec_{\mathcal{P}_s}} C^{\mathcal{P}_s}$ and $x \notin D^{\mathcal{P}_s}$, and therefore, we do not have $\mathcal{P}_s \Vdash C \precsim D$. Hence, it is not the case that $\mathcal{P}_s \Vdash \mathcal{KB}$, which contradicts our assumption. □

B PROOF OF THEOREM 1

Theorem 1 (Representation Result for Preferential Subsumption). *A defeasible subsumption relation* $\precsim\ \subseteq \mathcal{L} \times \mathcal{L}$ *is preferential if and only if there is a preferential interpretation* $\mathcal{P}$ *such that* $\precsim_{\mathcal{P}}\ =\ \precsim$.

B.1 If Part

We show that $\precsim_{\mathcal{P}}$ is preferential for every preferential interpretation $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec^{\mathcal{P}} \rangle$.

(Ref): Let $x \in \Delta^{\mathcal{P}}$ be such that $x \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$. Then clearly $x \in C^{\mathcal{P}}$, and therefore $\mathcal{P} \Vdash C \precsim C$. Hence, $C \precsim_{\mathcal{P}} C$.

(LLE): Assume that $C \precsim_{\mathcal{P}} E$ and $\mathcal{P} \Vdash C \equiv D$. Then $\mathcal{P} \Vdash C \precsim E$, which means $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq E^{\mathcal{P}}$. Since $\mathcal{P} \Vdash C \equiv D$, i.e., $C^{\mathcal{P}} = D^{\mathcal{P}}$, we have $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} = \min_{\prec^{\mathcal{P}}} D^{\mathcal{P}}$. Hence, $\min_{\prec^{\mathcal{P}}} D^{\mathcal{P}} \subseteq E^{\mathcal{P}}$, and therefore $\mathcal{P} \Vdash D \precsim E$, from which follows $D \precsim_{\mathcal{P}} E$.

(And): Assume we have both $C \precsim_{\mathcal{P}} D$ and $C \precsim_{\mathcal{P}} E$. Then $\mathcal{P} \Vdash C \precsim D$ and $\mathcal{P} \Vdash C \precsim E$, i.e., $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$ and $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq E^{\mathcal{P}}$, and then $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}} \cap E^{\mathcal{P}}$, from which follows $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq (D \sqcap E)^{\mathcal{P}}$. Hence, $\mathcal{P} \Vdash C \precsim D \sqcap E$, and therefore $C \precsim_{\mathcal{P}} D \sqcap E$.

(Or): Assume we have both $C \precsim_{\mathcal{P}} E$ and $D \precsim_{\mathcal{P}} E$. Let $x \in \min_{\prec^{\mathcal{P}}} (C \sqcup D)^{\mathcal{P}}$. Then x is minimal in $C^{\mathcal{P}} \cup D^{\mathcal{P}}$, and therefore either $x \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$ or $x \in \min_{\prec^{\mathcal{P}}} D^{\mathcal{P}}$. In either case $x \in E^{\mathcal{P}}$. Hence, $\mathcal{P} \Vdash C \sqcup D \precsim E$, and therefore $C \sqcup D \precsim_{\mathcal{P}} E$.

(RW): Assume we have both $C \precsim_{\mathcal{P}} D$ and $\mathcal{P} \Vdash D \sqsubseteq E$. Then $\mathcal{P} \Vdash C \precsim D$, i.e., $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$, and $D^{\mathcal{P}} \subseteq E^{\mathcal{P}}$. Hence, $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq E^{\mathcal{P}}$ and then $\mathcal{P} \Vdash C \precsim E$. Therefore, $C \precsim_{\mathcal{P}} E$.

(CM): Assume we have both $C \precsim_{\mathcal{P}} D$ and $C \precsim_{\mathcal{P}} E$. Then $\mathcal{P} \Vdash C \precsim D$ and $\mathcal{P} \Vdash C \precsim E$, and therefore $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$ and $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq E^{\mathcal{P}}$. Let $x \in \min_{\prec^{\mathcal{P}}} (C \sqcap D)^{\mathcal{P}}$. We show that $x \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$. Suppose this is not the case. Since $\prec^{\mathcal{P}}$ is smooth, there must be $x' \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$ such that $x' \prec^{\mathcal{P}} x$. Because $\mathcal{P} \Vdash C \precsim D$, $x' \in D^{\mathcal{P}}$, and then $x' \in C^{\mathcal{P}} \cap D^{\mathcal{P}}$, i.e., $x' \in (C \sqcap D)^{\mathcal{P}}$. From this and $x' \prec^{\mathcal{P}} x$ it follows that x is not minimal in $(C \sqcap D)^{\mathcal{P}}$, which is a contradiction. Hence, $x \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$. From this and $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq E^{\mathcal{P}}$, it follows that $x \in E^{\mathcal{P}}$. Hence, $\mathcal{P} \Vdash C \sqcap D \precsim E$, and therefore $C \sqcap D \precsim_{\mathcal{P}} E$.

B.2 Only-if Part

NB: The results marked $(*)$ are introduced here in the Appendix, while they are omitted in the main text.

Let $\precsim\ \subseteq \mathcal{L} \times \mathcal{L}$ be a preferential subsumption relation. We shall construct a preferential interpretation $\mathcal{P}$ such that $\precsim_{\mathcal{P}} =_{\text{def}} \{(C, D) \mid \mathcal{P} \Vdash C \precsim D\} = \precsim$.

Definition 29. Let $\mathscr{U} =_{\text{def}} \{(\mathcal{I}, x) \mid \mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle \text{ and } x \in \Delta^{\mathcal{I}}\}$.

Intuitively, $\mathscr{U}$ denotes the *universe* of objects in the context of their respective DL interpretations, i.e., $\mathscr{U}$ is a set of first-order interpretations.

Definition 30. A pair $(\mathcal{I}, x) \in \mathscr{U}$ is **normal** for $C \in \mathcal{L}$ if for every $D \in \mathcal{L}$ such that $C \precsim D$, we have $x \in D^{\mathcal{I}}$.

Lemma 16 $(*)$. *Let* $\precsim\ \subseteq \mathcal{L} \times \mathcal{L}$ *satisfy (Ref), (RW) and (And), and let* $C, D \in \mathcal{L}$. *Then all normal* $(\mathcal{I}, x)$ *for* C *satisfy* D *if and only if* $C \precsim D$.

Proof. The if part follows from the definition of normality above. For the only-if part, assume we do not have $C \precsim D$. We build a pair $(\mathcal{I}, x)$ that is normal for C but that does not satisfy D. Let $\Gamma =_{\text{def}} \{\neg D\} \cup \{E \in \mathcal{L} \mid C \precsim E\}$. All we need to do is show that there is $(\mathcal{I}, x)$ such that $x \in F^{\mathcal{I}}$ for every $F \in \Gamma$. Suppose this is not the case. Then by compactness there exists a finite $\Gamma' \subseteq \Gamma$ such that $\models \bigsqcap_{F \in \Gamma'} F \sqsubseteq D$. From this follows $\models \top \sqsubseteq \neg \bigsqcap_{F \in \Gamma'} F \sqcup D$, and, in particular, we have $\models C \sqsubseteq \neg \bigsqcap_{F \in \Gamma'} F \sqcup D$. Now from (Ref), we have $C \precsim C$. From this, $\models C \sqsubseteq \neg \bigsqcap_{F \in \Gamma'} F \sqcup D$, and (RW) we get $C \precsim (\neg \bigsqcap_{F \in \Gamma'} F \sqcup D)$. But, we also have $C \precsim \bigsqcap_{F \in \Gamma'} F$ by the (And) rule, and then by applying (And), once more we derive $C \precsim \bigsqcap_{F \in \Gamma'} F \sqcap (\neg \bigsqcap_{F \in \Gamma'} F \sqcup D)$. From this and (RW), we conclude $C \precsim D$, from which we derive a contradiction. □

Lemma 17 (∗). *If $\precsim$ is preferential, then the following rule holds:*

$$\frac{C \sqcup D \precsim C,\ D \sqcup E \precsim D}{C \sqcup E \precsim C}.$$

Proof. The proof is analogous to that by Kraus et al. [67, Lemma 22]. □

Definition 31. Let $C, D \in \mathcal{L}$. $C \leq D$ if $C \sqcup D \precsim C$.

Lemma 18 (∗). *If $\precsim$ is preferential, then $\leq$ is reflexive and transitive.*

Proof. From (Ref), we have $C \precsim C$. This and (LLE) gives us $C \sqcup C \precsim C$, therefore, we have $C \leq C$ and $\leq$ is reflexive. Transitivity follows from Lemma 17. □

Lemma 19 (∗). *If $\precsim$ is preferential, then the following rule holds:*

$$\frac{C \sqcup D \precsim C,\ D \precsim E}{C \precsim \neg D \sqcup E}.$$

Proof. The proof is analogous to that by Kraus et al. [67, Lemma 5.5]. □

Lemma 20 (∗). *If $C \leq D$ and $(\mathcal{I}, x)$ is normal for C, and $x \in D^{\mathcal{I}}$, then $(\mathcal{I}, x)$ is normal for D.*

Proof. From $C \leq D$, we get $C \sqcup D \precsim C$. Assume that $D \precsim E$ is the case. Then by Lemma 19, we have $C \precsim \neg D \sqcup E$. Since $(\mathcal{I}, x)$ is normal for C, we have $x \in (\neg D \sqcup E)^{\mathcal{I}}$. Given that $x \in D^{\mathcal{I}}$, we must have $x \in E^{\mathcal{I}}$. □

Lemma 21 (∗). *If $\precsim$ is preferential, then the following rule holds:*

$$\frac{C \sqcup D \precsim C,\ D \sqcup E \precsim D}{C \precsim \neg E \sqcup D}.$$

Proof. The proof is analogous to that by Kraus et al. [67, Lemma 5.5]. □

Lemma 22 (∗). *If $C \leq D \leq E$ and $(\mathcal{I}, x)$ is normal for C, and $x \in E^{\mathcal{I}}$, then $(\mathcal{I}, x)$ is normal for D.*

Proof. By Lemma 20, it is enough to show that $x \in D^{\mathcal{I}}$. By Lemma 21, we have $C \precsim \neg E \sqcup D$. Since $(\mathcal{I}, x)$ is normal for C and $x \in E^{\mathcal{I}}$, then we must have $x \in D^{\mathcal{I}}$. □

We now construct a preferential interpretation as in Definition 5.

Let $\mathcal{C}_{\perp} =_{\text{def}} \{C \mid C \precsim \perp\}$ and let $\mathscr{I} =_{\text{def}} \{\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle \mid C^{\mathcal{I}} = \emptyset \text{ for all } C \in \mathcal{C}_{\perp}\}$. Intuitively, $\mathscr{I}$ contains all interpretations that are "compatible" with $\precsim$ in the sense of not satisfying concepts that are defeasibly subsumed by the contradiction.

For each $\mathcal{I} \in \mathscr{I}$, let $\mathcal{I}^{+} =_{\text{def}} \langle \Delta^{\mathcal{I}^{+}}, \cdot^{\mathcal{I}^{+}} \rangle$ be such that:

- $\Delta^{\mathcal{I}^{+}} =_{\text{def}} X^{C} \cup X^{\perp}$, where $X^{C} =_{\text{def}} \{\langle \mathcal{I}, x, C \rangle \mid (\mathcal{I}, x) \text{ is normal for } C \in \mathcal{L}\}$, and $X^{\perp} =_{\text{def}} \{\langle \mathcal{I}, x, \perp \rangle \mid (\mathcal{I}, x) \text{ is not normal for any } C \in \mathcal{L}\}$;

- $\cdot^{\mathcal{I}^+}$ is such that for every $D \in \mathcal{L}$, $\langle \mathcal{I}, x, C\rangle \in D^{\mathcal{I}^+}$ if and only if $x \in D^{\mathcal{I}}$, and for every $r \in \mathsf{R}$, $(\langle \mathcal{I}, x, C\rangle, \langle \mathcal{I}, y, D\rangle) \in r^{\mathcal{I}^+}$ if and only if $(x, y) \in r^{\mathcal{I}}$.

Let $\mathcal{P} =_{\text{def}} \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec^{\mathcal{P}}\rangle$ be such that:

- $\Delta^{\mathcal{P}} =_{\text{def}} \bigcup_{\mathcal{I} \in \mathscr{I}} \Delta^{\mathcal{I}^+}$;
- $\cdot^{\mathcal{P}} =_{\text{def}} \bigcup_{\mathcal{I} \in \mathscr{I}} \cdot^{\mathcal{I}^+}$;
- $\prec^{\mathcal{P}}$ is the smallest relation such that:
  - For every $\langle \mathcal{I}, x, C\rangle \in \Delta^{\mathcal{P}}$ such that $C \neq \bot$, $\langle \mathcal{I}, x, C\rangle \prec^{\mathcal{P}} \langle \mathcal{J}, y, \bot\rangle$ for every $\langle \mathcal{J}, y, \bot\rangle \in \Delta^{\mathcal{P}}$;
  - For every $\langle \mathcal{I}, x, C\rangle, \langle \mathcal{J}, y, D\rangle \in \Delta^{\mathcal{P}}$ such that $C, D \neq \bot$, $\langle \mathcal{I}, x, C\rangle \prec^{\mathcal{P}} \langle \mathcal{J}, y, D\rangle$ if and only if $C \leq D$ and $x \notin D^{\mathcal{I}}$.

(In the construction of $\mathcal{P}$, note that all pairs $(\mathcal{I}, x)$ that are not normal for any concept C are moved higher up in the ordering so that they correspond to the least preferred objects of the domain.)

In Lemmas 23 to 28, below we show that $\mathcal{P}$ as constructed above is indeed a preferential interpretation.

Lemma 23 (∗). *$\Delta^{\mathcal{P}} \neq \emptyset$.*

Proof. From the fact that $\top \precsim \bot$ does not hold and Lemma 16, it follows that there is some normal $(\mathcal{I}, x)$ for $\top$ that does not satisfy $\bot$. Hence, $\langle \mathcal{I}, x, \top\rangle \in \Delta^{\mathcal{P}}$, and therefore $\Delta^{\mathcal{P}} \neq \emptyset$. □

Lemma 24 (∗). *$C \leq \bot$ for every $C \in \mathcal{L}$.*

Proof. By (Ref), we have $C \precsim C$. Since $\models C \equiv C \sqcup \bot$, by (LLE), we get $C \sqcup \bot \precsim C$, and then from the definition of $\leq$ follows $C \leq \bot$. □

Lemma 25 (∗). *$\prec^{\mathcal{P}}$ is a strict partial order on $\Delta^{\mathcal{P}}$, i.e., $\prec^{\mathcal{P}}$ is irreflexive and transitive.*

Proof. First, we show irreflexivity. From the construction of $\prec^{\mathcal{P}}$, it clearly follows that for every $\langle \mathcal{I}, x, \bot\rangle \in \Delta^{\mathcal{P}}$, $(\langle \mathcal{I}, x, \bot\rangle, \langle \mathcal{I}, x, \bot\rangle) \notin \prec$. Assume that $\langle \mathcal{I}, x, C\rangle \prec^{\mathcal{P}} \langle \mathcal{I}, x, C\rangle$ for some $C \neq \bot$. Then $C \leq C$ and $x \notin C^{\mathcal{I}}$, i.e., $C \sqcup C \precsim C$, and then $C \precsim C$, by (LLE). This and $x \notin C^{\mathcal{I}}$ contradicts the fact that $(\mathcal{I}, x)$ is normal for C. Hence, $(\langle \mathcal{I}, x, C\rangle, \langle \mathcal{I}, x, C\rangle) \notin \prec^{\mathcal{P}}$ for every $\langle \mathcal{I}, x, C\rangle \in \Delta^{\mathcal{I}}$.

We now show transitivity. Suppose $\langle \mathcal{I}, x, C\rangle \prec^{\mathcal{P}} \langle \mathcal{I}', x', D\rangle$ and $\langle \mathcal{I}', x', D\rangle \prec^{\mathcal{P}} \langle \mathcal{I}'', x'', E\rangle$. From the definition of $\prec^{\mathcal{P}}$, we know that $C, D \neq \bot$, since all non-normal objects are at the highest level in the ordering and are all incomparable. We then have $C \leq D$ and $D \leq E$. (If $E = \bot$, then we also have $D \leq E$ by Lemma 24.) From transitivity of $\leq$ (Lemma 18), we conclude $C \leq E$. Since $\langle \mathcal{I}, x, C\rangle \in \Delta^{\mathcal{P}}$ and $\langle \mathcal{I}, x, C\rangle \prec^{\mathcal{P}} \langle \mathcal{I}', x', D\rangle$, we conclude that $(\mathcal{I}, x)$ is normal for C and $x \notin D^{\mathcal{P}}$. This and Lemma 22 imply that $x \notin E^{\mathcal{P}}$. □

Lemma 26 (∗). *Given $\langle \mathcal{I}, x, D\rangle \in \Delta^{\mathcal{P}}$, $\langle \mathcal{I}, x, D\rangle \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$ iff $x \in C^{\mathcal{I}}$ and $D \leq C$.*

Proof. For the if part, suppose that $x \in C^{\mathcal{I}}$ and $D \leq C$. Then it clearly follows that $\langle \mathcal{I}, x, D\rangle \in C^{\mathcal{P}}$ (Lemma 20). Now suppose that $\langle \mathcal{I}, x, D\rangle$ is not $\prec^{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$, i.e., there is $\langle \mathcal{I}', x', E\rangle$ for some $\mathcal{I}'$ such that $x' \in \Delta^{\mathcal{I}'}$ and some $E \in \mathcal{L}$ such that $\langle \mathcal{I}', x', E\rangle \prec^{\mathcal{P}} \langle \mathcal{I}, x, D\rangle$ and $x' \in C^{\mathcal{I}'}$. From this and the definition of $\prec^{\mathcal{P}}$, it follows that $E \leq D$ and $x' \notin D^{\mathcal{I}'}$. Hence, $E \leq D \leq C$ and $(\mathcal{I}', x')$ is normal for E, and since $x' \in C^{\mathcal{I}'}$, by Lemma 22, we get that $(\mathcal{I}', x')$ is normal for D, from which we conclude $x' \in D^{\mathcal{I}'}$, a contradiction.

For the only-if part, suppose that $\langle \mathcal{I}, x, D\rangle$ is $\prec^{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. Then clearly $x \in C^{\mathcal{I}}$. Now assume there is some $(\mathcal{I}', x')$, which is normal for $C \sqcup D$ and $x' \notin D^{\mathcal{I}'}$. Since $C \sqcup D \leq D$, we must have $\langle \mathcal{I}', x', C \sqcup D\rangle \prec^{\mathcal{P}} \langle \mathcal{I}, x, D\rangle$. Since $(\mathcal{I}', x')$ is normal for $C \sqcup D$ and $x' \notin D^{\mathcal{I}'}$, it follows that $x' \in C^{\mathcal{I}'}$. This contradicts the minimality of $\langle \mathcal{I}, x, D\rangle$ in $C^{\mathcal{P}}$. Hence, all normal $(\mathcal{I}', x')$ for $C \sqcup D$ satisfy D. From this and Lemma 16 follows $C \sqcup D \precsim D$, i.e., $D \leq C$. □

Lemma 27 (∗). *There is no $C \in \mathcal{L}$ such that $C^{\mathcal{P}} \neq \emptyset$ and $\bot \leq C$.*

Proof. Let $C \in \mathcal{L}$ be such that $C^{\mathcal{P}} \neq \emptyset$. Assume that $\bot \leq C$. Then, $\bot \sqcup C \precsim \bot$, i.e., $C \precsim \bot$. Then, $C \in \mathcal{C}_{\bot}$, and then $C^{\mathcal{P}} = \emptyset$ by the construction of $\mathcal{P}$. □

Corollary 5 (∗). *It follows from the two last lemmas that there is no $C \in \mathcal{L}$ for which any $\langle \mathcal{I}, x, \bot \rangle \in \Delta^{\mathcal{P}}$ is minimal.*

Lemma 28 (∗). *For any $C \in \mathcal{L}$, $C^{\mathcal{P}}$ is smooth.*

Proof. Suppose that $\langle \mathcal{I}, x, D \rangle \in C^{\mathcal{P}}$, i.e., $x \in C^{\mathcal{I}}$. If $D \leq C$, then by Lemma 26 $\langle \mathcal{I}, x, D \rangle$ is $\prec^{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. However, i.e., if $D \not\leq C$, then we do not have $C \sqcup D \precsim D$, then by Lemma 16 there is a normal $(\mathcal{I}', x')$ for $C \sqcup D$ such that $x \notin D^{\mathcal{I}'}$. But $C \sqcup D \precsim C \sqcup D$, and then $(C \sqcup D) \sqcup D \precsim C \sqcup D$, and then $C \sqcup D \leq D$. Hence, $\langle \mathcal{I}', x', C \sqcup D \rangle \prec^{\mathcal{P}} \langle \mathcal{I}, x, D \rangle$. But $x' \in (C \sqcup D)^{\mathcal{I}'}$ and $x' \notin D^{\mathcal{I}'}$, therefore $x' \in C^{\mathcal{I}'}$. Since $C \sqcup D \leq C$, from Lemma 26, we conclude that $\langle \mathcal{I}', x', C \sqcup D \rangle$ is $\prec^{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. □

Next, we show in Lemma 29 that the abstract relation $\precsim$ we started off with coincides with the relation $\precsim_{\mathcal{P}}$ obtained from our constructed preferential interpretation $\mathcal{P}$.

Lemma 29 (∗). *$C \precsim D$ if and only if $C \precsim_{\mathcal{P}} D$.*

Proof. For the only-if part, we show that $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$. Let $\langle \mathcal{I}, x, E \rangle$ be $\prec^{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. Then $(\mathcal{I}, x)$ is normal for E and $x \in C^{\mathcal{P}}$, and from Lemma 26, we also have $E \leq C$. From these results and Lemma 20 it follows that $(\mathcal{I}, x)$ is normal for C. Since $C \precsim D$, we have $x \in D^{\mathcal{I}}$, and therefore $\langle \mathcal{I}, x, E \rangle \in \Delta^{\mathcal{P}}$.

For the if part, let $C \precsim_{\mathcal{P}} D$. From the definition of $\prec^{\mathcal{P}}$, it follows that for every $(\mathcal{I}, x)$ normal for C, $\langle \mathcal{I}, x, C \rangle \in \min_{\prec^{\mathcal{P}}} C^{\mathcal{P}}$. Since $C \precsim_{\mathcal{P}} D$, then $y \in D^{\mathcal{I}'}$ for every $(\mathcal{I}', y)$ that is normal for C. This and Lemma 16 give us $C \precsim D$. □

Proof of Theorem 1:
Soundness, the if part, is given in Section B.1. For the only-if part, let $\precsim$ be a preferential subsumption relation and let $\mathcal{P}$ be defined as above. Lemmas 23, 25 and 28 show that $\mathcal{P}$ is a preferential DL interpretation. Lemma 29 shows that $\mathcal{P}$ defines a subsumption relation that is exactly $\precsim$.

C PROOF OF THEOREM 2

Theorem 2 (Representation Result for Rational Subsumption). *A defeasible subsumption relation $\precsim \subseteq \mathcal{L} \times \mathcal{L}$ is rational if and only if there is a modular interpretation $\mathcal{R}$ such that $\precsim_{\mathcal{R}} = \precsim$.*

C.1 If Part

Satisfaction of the basic KLM properties for preferential subsumption follows from the proof in Section B.1, given the fact that modular interpretations are a special case of preferential interpretations. Below, we show that rational monotonicity is satisfied.

Assume that $C \precsim_{\mathcal{R}} E$ and that we do not have $C \precsim_{\mathcal{R}} \neg D$. From the latter it follows that there is $x \in \min_{\prec^{\mathcal{R}}} C^{\mathcal{R}}$ such that $x \in D^{\mathcal{R}}$, i.e., $x \in (C \sqcap D)^{\mathcal{R}}$. Let now $x' \in \min_{\prec^{\mathcal{R}}} (C \sqcap D)^{\mathcal{R}}$. Since $x \in (C \sqcap D)^{\mathcal{R}}$, $(x, x') \notin \prec^{\mathcal{R}}$. This means that $x' \in \min_{\prec^{\mathcal{R}}} C^{\mathcal{R}}$, for if there is $x'' \in C^{\mathcal{R}}$ such that $x'' \prec^{\mathcal{R}} x'$, then $x'' \prec^{\mathcal{R}} x$, which is impossible, since x is minimal in $C^{\mathcal{R}}$. From $x' \in \min_{\prec^{\mathcal{R}}} C^{\mathcal{R}}$ and $\mathcal{R} \Vdash C \precsim E$ follows $x' \in E^{\mathcal{R}}$. Hence, $\mathcal{R} \Vdash C \sqcap D \precsim E$, and therefore $C \sqcap D \precsim_{\mathcal{R}} E$.

C.2 Only-if Part

NB: The results marked (∗) are introduced here in the Appendix, while they are omitted in the main text.

The proof of the only-if part relies on the results for the preferential case (Section B.1), with the main difference being the definition of the preference relation, which is shown to be a smooth modular order. This ensures that the canonical model constructed in the proof is a modular interpretation.

Let $\precsim \subseteq \mathcal{L} \times \mathcal{L}$ satisfy all the basic properties of preferential subsumption relations together with rational monotonicity.

The proof of the following lemma is analogous to that of Lemma 3.11 by Lehmann and Magidor [70]:

Lemma 30 (∗). *If* $\precsim$ *is rational, then the properties below hold:*

$$\frac{C \sqcup D \precsim \neg D}{C \precsim \neg D}, \qquad \frac{C \sqcup E \precsim \neg C,\ D \sqcup E \not\precsim \neg D}{C \sqcup D \precsim \neg C}.$$

Definition 32. Let $C \in \mathcal{L}$. We say that C is **consistent** w.r.t. $\precsim$ if $C \not\precsim \bot$. Given $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, we say that C is **consistent** w.r.t. $\precsim_{\mathcal{R}}$ if $C \precsim_{\mathcal{R}} \bot$ does not hold, i.e., if there is $x \in \Delta^{\mathcal{R}}$ s.t. $x \in C^{\mathcal{R}}$.

Let $\mathcal{C} = \{C \in \mathcal{L} \mid C \text{ is consistent w.r.t. } \precsim\}$.

Lemma 31 (∗). *Let* $C \in \mathcal{L}$ *and let* $\precsim$ *be a rational relation. Then* $C \in \mathcal{C}$ *iff there is* $(\mathcal{I}, x) \in \mathscr{U}$ *s.t.* $(\mathcal{I}, x)$ *is normal for* C*. (Cf. Definitions 29 and 30 in Appendix B.2.)*

Definition 33. Given $C, D \in \mathcal{C}$, C is **not more exceptional than** D, written $C\mathscr{R}D$, if $C \sqcup D \not\precsim \neg C$. We say that C is **as exceptional as** D, written $C \sim D$, if $C\mathscr{R}D$ and $D\mathscr{R}C$.

The proof of the lemma below follows those of Lemmas A.4 and A.5 by Lehmann and Magidor [70]:

Lemma 32 (∗). $\mathscr{R}$ *is reflexive and transitive.*

That $\sim$ is an equivalence relation follows from the fact that $\mathscr{R}$ is reflexive and transitive (Lemma 32). With $[C]$, we denote the equivalence class of C. The set of equivalence classes of concepts of $\mathcal{C}$ under $\sim$ is denoted by $[\mathcal{C}]$. We write $[C] \leq [D]$ if $C\mathscr{R}D$, and $[C] < [D]$ if $[C] \leq [D]$ and $C \nsim D$.

Thanks to Lemma 32, we can state the following:

Lemma 33 (∗). *The relation* $<$ *is a strict order on* $[\mathcal{C}]$*.*

Lemma 34 (∗). *Let* $C, D \in \mathcal{L}$ *be consistent w.r.t.* $\precsim$*. If* $[C] < [D]$*, then* $C \precsim \neg D$*.*

Proof. The assumption implies that $C\mathscr{R}D$ is not the case, i.e., $C \sqcup D \precsim \neg C$. This and Lemma 30 imply the conclusion. □

Lemma 34 warrants the following result:

Lemma 35 (∗). *Let* $C, D \in \mathcal{L}$ *be consistent w.r.t.* $\precsim$*. If there is* $(\mathcal{I}, x) \in \mathscr{U}$ *s.t.* $(\mathcal{I}, x)$ *is normal for* C *and* $x \in D^{\mathcal{I}}$*, then* $[D] \leq [C]$*.*

Armed with these results, we can then construct an interpretation $\mathcal{R}$ analogous to the preferential interpretation $\mathcal{P}$ in Appendix B.2, with the preference relation defined as follows:

- For every $\langle \mathcal{I}, x, C \rangle \in \Delta^{\mathcal{R}}$ such that $C \neq \bot$, $\langle \mathcal{I}, x, C \rangle \prec^{\mathcal{R}} \langle \mathcal{J}, y, \bot \rangle$ for every $\langle \mathcal{J}, y, \bot \rangle \in \Delta^{\mathcal{R}}$;
- For every $\langle \mathcal{I}, x, C \rangle, \langle \mathcal{J}, y, D \rangle \in \Delta^{\mathcal{R}}$ such that $C, D \neq \bot$, $\langle \mathcal{I}, x, C \rangle \prec^{\mathcal{R}} \langle \mathcal{J}, y, D \rangle$ if $[C] < [D]$.

It is not hard to see that this definition implies the following result:

Lemma 36 (∗). $\prec^{\mathcal{R}}$ *is a modular partial order.*

The proof of the following lemma follows that of Lehmann and Magidor's Lemma A.12 [70]:

Lemma 37 (∗). *For every $C \in \mathcal{L}$, if C is consistent, then $C^{\mathcal{R}}$ is smooth.*

From this point on, a result analogous to Lemma 29 in B.2 can be shown to hold for the defeasible subsumption $\precsim_{\mathcal{R}}$ induced by $\mathcal{R}$. From that the result follows.

D PROOFS OF RESULTS IN SECTION 4

Lemma 4. *Lemma Let $\mathcal{KB}$ be a defeasible knowledge base. Then,*

$$\precsim_{\mathcal{KB}^*_{\text{pref}}} = \bigcap\{\precsim_{\mathcal{K}} \mid \mathcal{KB} \subseteq \mathcal{K} \text{ and } \mathcal{K} \text{ is preferential}\}.$$

Proof. By Definitions 10 and 11, $\alpha \in \mathcal{KB}^*_{\text{pref}}$ iff for every preferential model $\mathcal{P}$ of $\mathcal{KB}$, $\mathcal{P} \Vdash \alpha$. Combined with Lemma 2, this implies that, for any defeasible subsumption $C \precsim D$, $C \precsim D \in \mathcal{KB}^*_{\text{pref}}$ iff $(C, D) \in \precsim_{\mathcal{P}}$ for every preferential model $\mathcal{P}$ of $\mathcal{KB}$. Due to Theorem 1, the latter condition, that is, $(C, D) \in \precsim_{\mathcal{P}}$ for every preferential model $\mathcal{P}$ of $\mathcal{KB}$, holds iff $(C, D) \in \precsim_{\mathcal{K}}$ for every preferential theory $\mathcal{K}$ containing $\mathcal{KB}$. This concludes the proof. □

Lemma 5. *Let $\mathcal{KB}$ be a defeasible knowledge base. Then,*

$$\precsim_{\mathcal{KB}^*_{\text{mod}}} = \bigcap\{\precsim_{\mathcal{K}} \mid \mathcal{KB} \subseteq \mathcal{K} \text{ and } \mathcal{K} \text{ is rational}\}.$$

Proof. The proof follows the one of Lemma 4. It is sufficient to refer to Definitions 14 and 15 instead of Definitions 10 and 11, and to Theorem 2 instead of Theorem 1. □

Lemma 7. *A modular interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ s.t. $\Delta^{\mathcal{R}}$ is finite is a ranked interpretation.*

Proof. The preference relation $\prec^{\mathcal{R}}$ is a strict partial order, hence, since there cannot be cycles, for every finite set $\emptyset \neq X \subseteq \Delta^{\mathcal{R}}$, $\min_{\prec^{\mathcal{R}}} X \neq \emptyset$. We can define the function $h_{\mathcal{R}}(\cdot)$ in the following way:

(1) $\Delta^{\mathcal{R}^0} =_{\text{def}} \Delta^{\mathcal{R}}$;
(2) $i =_{\text{def}} 0$;
(3) If $\Delta^{\mathcal{R}^i} \neq \emptyset$, then proceed; else, return the function $h_{\mathcal{R}}$;
(4) $h_{\mathcal{R}}(x) = i$ iff $x \in \min_{\prec^{\mathcal{R}}} \Delta^{\mathcal{R}^i}$; let $\Delta^{\mathcal{R}^{i+1}} =_{\text{def}} \Delta^{\mathcal{R}^i} \setminus \min_{\prec^{\mathcal{R}}} \Delta^{\mathcal{R}^i}$;
(5) $i =_{\text{def}} i + 1$;
(6) Go back to step 3.

It is easy to check that $h_{\mathcal{R}}(\cdot)$ satisfies the convexity property and characterises $\prec_{\mathcal{R}}$ (i.e., $x \prec_{\mathcal{R}} y$ iff $h_{\mathcal{R}}(x) < h_{\mathcal{R}}(y)$). □

Proposition 1. *Given a ranked interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, there is only one function $h_{\mathcal{R}} : X \longrightarrow \mathbb{N}$ satisfying the convexity property and s.t. for every $x, y \in X$, $x \prec y$ iff $h_{\mathcal{R}}(x) < h_{\mathcal{R}}(y)$.*

Proof. Assume that for a ranked interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ there are two distinct functions $h_{\mathcal{R}}(\cdot)$ and $h'_{\mathcal{R}}(\cdot)$ satisfying the convexity constraint and characterising $\prec^{\mathcal{R}}$. Since the two functions are distinct, at a certain point they must diverge; that is, there must be an $i \in \mathbb{N}$ s.t. for every $k < i$ and every $x \in \Delta^{\mathcal{R}}$, $h_{\mathcal{R}}(x) = k$ iff $h'_{\mathcal{R}}(x) = k$, but there is a $y \in \Delta^{\mathcal{R}}$ s.t. $h_{\mathcal{R}}(y) = i$ and $h'_{\mathcal{R}}(y) = j$, with $j > i$. The convexity constraint imposes that there must be a $z \in \Delta^{\mathcal{R}}$ s.t. $h'_{\mathcal{R}}(z) = i$: then $h'_{\mathcal{R}}(\cdot)$ enforces $z \prec^{\mathcal{R}} y$, while according to $h_{\mathcal{R}}(\cdot)$ that cannot be the case (it must be $h_{\mathcal{R}}(y) \leq h_{\mathcal{R}}(z)$). □

Some extra material needs to be introduced to prove Theorem 3, stating the Finite Model Property for Defeasible $\mathcal{ALC}$. First, we will refer to the following semantic construction.

Definition 34 (Finite Model Construction). (∗) Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be a finite defeasible knowledge base, and let $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ be a modular model of $\mathcal{KB}$ (with $\Delta^{\mathcal{R}}$ possibly infinite). Let C, R be the sets of names of our language, as from Section 2, and Γ be the set of concepts $\{C_1, \dots, C_n\} \subseteq \mathcal{L}$ obtained by closing the set of all concepts appearing in the axioms in $\mathcal{KB}$ under sub-concepts and negation. We define the equivalence relation $\approx_\Gamma$ as follows: for every $x, y \in \Delta^{\mathcal{R}}$, $x \approx_\Gamma y$ if for every $C \in \Gamma$, $x \in C^{\mathcal{R}}$ iff $y \in C^{\mathcal{R}}$.

We indicate with $[x]_\Gamma$ the equivalence class of the objects that are related to an object x through $\approx_\Gamma$:

$$[x]_\Gamma =_{\text{def}} \{y \in \Delta^{\mathcal{R}} \mid x \approx_\Gamma y\}.$$

We introduce a new model $\mathcal{R}' = \langle \Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}'}, \prec^{\mathcal{R}'} \rangle$, defined as follows:

- $\Delta^{\mathcal{R}'} = \{[x]_\Gamma \mid x \in \Delta^{\mathcal{R}}\}$;
- For every $A \in \mathsf{C} \cap \Gamma$, $A^{\mathcal{R}'} = \{[x]_\Gamma \mid x \in A^{\mathcal{R}}\}$;
- For every $A \notin \mathsf{C} \cap \Gamma$, $A^{\mathcal{R}'} = \emptyset$;
- For every $r \in \mathsf{R}$, $r^{\mathcal{R}'} = \{([x]_\Gamma, [y]_\Gamma) \mid (x, y) \in r^{\mathcal{R}}\}$;
- For every $[x]_\Gamma, [y]_\Gamma \in \Delta^{\mathcal{R}'}$, $[x]_\Gamma \prec^{\mathcal{R}'} [y]_\Gamma$ if there is an object $z \in [x]_\Gamma$ s.t. for all the objects $v \in [y]_\Gamma$, $z \prec^{\mathcal{R}} v$;

Let $\sim^{\mathcal{R}'}$ be the indifference relation, defined as usual:

- $[x]_\Gamma \sim^{\mathcal{R}'} [y]_\Gamma$ if $([x]_\Gamma, [y]_\Gamma) \notin \prec^{\mathcal{R}'}$ and $([y]_\Gamma, [x]_\Gamma) \notin \prec^{\mathcal{R}'}$.

Given that Γ is finite, $\Delta^{\mathcal{R}'}$ is clearly finite. The following results are easy to prove.

Lemma 38 (∗). *For every $C \in \Gamma$ and every $x \in \Delta^{\mathcal{R}}$, $x \in C^{\mathcal{R}}$ iff $[x]_\Gamma \in C^{\mathcal{R}'}$.*

Proof. The proof is analogous to that for the classical case and is by induction on the structure of concepts. □

Lemma 39 (∗). *Let $\prec^{\mathcal{R}'}$ and $\prec^{\mathcal{R}}$ be as in Definition 34. Then $\prec^{\mathcal{R}'}$ is a strict partial order.*

Proof. We show that $\prec^{\mathcal{R}'}$ is irreflexive and transitive.

Irreflexivity: Assume $[x]_\Gamma \prec^{\mathcal{R}'} [x]_\Gamma$. By the definition of $\prec^{\mathcal{R}'}$, it implies that there is a $z \in [x]_\Gamma$ s.t. $z \prec^{\mathcal{R}} v$ for every $v \in [x]_\Gamma$. That is, we have that $z \prec^{\mathcal{R}} z$ that, since $\prec^{\mathcal{R}}$ is a strict partial order (Definitions 5 and 9), cannot be the case.

Transitivity: Assume $[x]_\Gamma \prec^{\mathcal{R}'} [y]_\Gamma$ and $[y]_\Gamma \prec^{\mathcal{R}'} [u]_\Gamma$. This means that there is a $z \in [x]_\Gamma$ s.t. $z \prec^{\mathcal{R}} v$ for every $v \in [y]_\Gamma$, and there is a $v' \in [y]_\Gamma$ s.t. $v' \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$. Since $\prec^{\mathcal{R}}$ is transitive, it follows that there is a $z \in [x]_\Gamma$ s.t. $z \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$, that is, $[x]_\Gamma \prec^{\mathcal{R}'} [u]_\Gamma$. □

Lemma 40 (∗). *Let $\sim^{\mathcal{R}'}$ be as in Definition 34. Then relation $\sim^{\mathcal{R}'}$ is transitive.*

Proof. Let $[x]_\Gamma \sim^{\mathcal{R}'} [y]_\Gamma$, $[y]_\Gamma \sim^{\mathcal{R}'} [u]_\Gamma$, but $[x]_\Gamma \nsim^{\mathcal{R}'} [u]_\Gamma$. The latter implies that either $[x]_\Gamma \prec^{\mathcal{R}'} [u]_\Gamma$ or $[u]_\Gamma \prec^{\mathcal{R}'} [x]_\Gamma$; w.l.o.g. let us assume $[x]_\Gamma \prec^{\mathcal{R}'} [u]_\Gamma$. That is, there is a $z \in [x]_\Gamma$ s.t. $z \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$.

$[x]_\Gamma \sim^{\mathcal{R}'} [y]_\Gamma$ implies that $z \sim^{\mathcal{R}} v$ for some $v \in [y]_\Gamma$. Assume the latter does not hold, then for every $v \in [y]_\Gamma$ either $z \prec^{\mathcal{R}} v$ or $v \prec^{\mathcal{R}} z$. It cannot be that $z \prec^{\mathcal{R}} v$ for every $v \in [y]_\Gamma$, since that would imply $[x]_\Gamma \prec^{\mathcal{R}'} [y]_\Gamma$, so there must be some $v \in [y]_\Gamma$ s.t. $v \prec^{\mathcal{R}} z$. However the latter would also imply, due to the transitivity of $\prec^{\mathcal{R}}$, that there is a $v \in [y]_\Gamma$ s.t. $v \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$, that is, $[y]_\Gamma \prec^{\mathcal{R}'} [u]_\Gamma$, against the hypothesis that $[y]_\Gamma \sim^{\mathcal{R}'} [u]_\Gamma$. Consequently, $z \sim^{\mathcal{R}} v$ for some $v \in [y]_\Gamma$.

So there is a $z \in [x]_\Gamma$ s.t. $z \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$ and there is a $v \in [y]_\Gamma$ s.t. $v \sim^{\mathcal{R}} z$. That implies that $v \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$. To see it, assume that it not the case, that is, we have that for some $w' \in [u]_\Gamma$ either $w' \prec^{\mathcal{R}} v$ or $w' \sim^{\mathcal{R}} v$: In the former case, we would obtain $z \prec^{\mathcal{R}} v$, in the latter $z \sim^{\mathcal{R}} w'$, both taking us to contradiction. Hence, $v \prec^{\mathcal{R}} w$ for every $w \in [u]_\Gamma$, that is, $[y]_\Gamma \prec^{\mathcal{R}'} [u]_\Gamma$, against the hypothesis. □

Lemma 41 (∗). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be finite. If $\mathcal{KB}$ has a modular model, then it has a finite-ranked model.*

Proof. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be a finite defeasible knowledge base, $\mathcal{R}$ a model of $\mathcal{KB}$ and $\mathcal{R}'$ a finite interpretation constructed from $\mathcal{R}$ as in Definition 34. $\mathcal{R}'$ is a finite interpretation, and it is modular, since Lemmas 39 and 40 prove that $\prec^{\mathcal{R}'}$ satisfies Definition 8. Being $\mathcal{R}'$ a finite modular interpretation, it is a finite-ranked interpretation (Lemma 7).

It remains to prove that $\mathcal{R}'$ is a model of $\mathcal{KB}$. The proof that $\mathcal{R}'$ satisfies $\mathcal{T}$ is straightforward by Lemma 38. With regard to $\mathcal{D}$, let $C \precsim D \in \mathcal{D}$. Since $\mathcal{R}$ is a model of $\mathcal{D}$, either $C^{\mathcal{R}} = \emptyset$, or the height of $C \sqcap D$ in $\mathcal{R}$ is lower than the height of $C \sqcap \neg D$, that is, there is at least an object y in $(C \sqcap D)^{\mathcal{R}}$ s.t. for every object x in $(C \sqcap \neg D)^{\mathcal{R}}$, $y \prec^{\mathcal{R}} x$. Since C, D, and $\neg D$ are in Γ, the object $[y]_\Gamma \in \Delta^{\mathcal{R}'}$ (obtained from $y \in (C \sqcap D)^{\mathcal{R}}$) must be preferred to all the objects in $(C \sqcap \neg D)^{\mathcal{R}}$, that is, $[y]_\Gamma \prec^{\mathcal{R}'} [x]_\Gamma$ for every object $[x]_\Gamma$ s.t. $[x]_\Gamma \in (C \sqcap \neg D)^{\mathcal{R}'}$. Therefore, $\mathcal{R}' \Vdash C \precsim D$. □

Lemma 42 (∗). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be finite and $C, D \in \mathcal{L}$. If $\mathcal{KB}$ has a modular model that is a counter-model to $C \precsim D$, then it has a finite-ranked model that is a counter-model to $C \precsim D$.*

Proof. It is sufficient to apply the same construction defined for the finite-model property above. We just need to add C and D to the set Γ (and close Γ also under the subconcepts of C and D and their negations). If $\mathcal{R} \Vdash C \precsim D$ does not hold, then there is an object x s.t. $x \in (C \sqcap \neg D)^{\mathcal{R}}$ and $x \prec^{\mathcal{R}} y$ or $x \sim^{\mathcal{R}} y$ for every object y s.t. $y \in (C \sqcap D)^{\mathcal{R}}$. That implies that in $\mathcal{R}'$ $([y]_\Gamma, [x]_\Gamma) \notin \prec_{\mathcal{R}'}$, that is, $[x]_\Gamma \prec_{\mathcal{R}'} [y]_\Gamma$ or $[x]_\Gamma \sim_{\mathcal{R}'} [y]_\Gamma$ for every y s.t. $y \in (C \sqcap D)^{\mathcal{R}}$, and consequently, we do not have $\mathcal{R}' \Vdash C \precsim D$. □

Corollary 6 (∗). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be a finite defeasible knowledge base. If $\mathcal{KB}$ has a modular model $\mathcal{R}$, then for every $C \in \mathcal{L}$ s.t. $h_{\mathcal{R}}(C) = 0$ there is also a finite-ranked model $\mathcal{R}'$ of $\mathcal{KB}$ s.t. $h_{\mathcal{R}'}(C) = 0$.*

Proof. Given $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, a model $\mathcal{R}$ of $\mathcal{KB}$ and a concept C s.t. $h_{\mathcal{R}}(C) = 0$, a finite model $\mathcal{R}'$ satisfying the constraint above can be defined in the same way as the model $\mathcal{R}'$ from Definition 34. We just need to add C to the set Γ (and close Γ also under the subconcepts of C and their negations). To see that $\mathcal{R}'$ is a model of $\mathcal{KB}$, just go again through the proof of the finite-model property above, and check that the addition of C to Γ does not affect any of the above results.

Now, $h_{\mathcal{R}}(C) = 0$ implies that there is an object $x \in \Delta^{\mathcal{R}}$ s.t. $x \in C^{\mathcal{R}}$ and $h_{\mathcal{R}}(x) = 0$. Consider now $[x]_\Gamma$. By Lemma 38, $[x]_\Gamma \in C^{\mathcal{R}'}$. Since $h_{\mathcal{R}}(x) = 0$, for every $[y]_\Gamma \in \Delta^{\mathcal{R}'}$ it cannot be the case that there is an object $z \in [y]_\Gamma$ s.t. $z \prec^{\mathcal{R}} v$ for every $v \in [x]_\Gamma$; hence, the definition of $\prec^{\mathcal{R}'}$ implies that for every $[y]_\Gamma \in \Delta^{\mathcal{R}'}$, $([y]_\Gamma, [x]_\Gamma) \notin \prec^{\mathcal{R}'}$, that is, $h_{\mathcal{R}'}([x]_\Gamma) = 0$, that implies $h_{\mathcal{R}'}(C) = 0$. □

Now, we can prove Theorem 3.

Theorem 3 (Finite-Model Property). *Defeasible $\mathcal{ALC}$ has the finite-model property. In particular, every defeasible $\mathcal{ALC}$ knowledge base that has a modular model, has also a finite-ranked model.*

Proof. The result follows straightforwardly from Lemmas 41 and 42. □

Lemma 8. *Given a set of ranked models of a defeasible knowledge base $\mathcal{KB}$, their ranked union is itself a ranked model of $\mathcal{KB}$.*

Proof. Let $\mathfrak{R}$ be a set of ranked models of a defeasible knowledge base $\mathcal{KB}$, and let $\mathcal{R}^{\mathfrak{R}} =_{\text{def}} \langle \Delta^{\mathfrak{R}}, \cdot^{\mathfrak{R}}, \prec^{\mathfrak{R}} \rangle$ be its ranked union. We want to prove that also $\mathcal{R}^{\mathfrak{R}}$ is a ranked model of $\mathcal{KB}$, and to do that is sufficient to prove that for every DCI $C \mathrel{\tilde{\sqsubseteq}} D$, if $\mathcal{R} \Vdash C \mathrel{\tilde{\sqsubseteq}} D$ for every $\mathcal{R} \in \mathfrak{R}$, then $\mathcal{R}^{\mathfrak{R}} \Vdash C \mathrel{\tilde{\sqsubseteq}} D$.

It is easy to prove by induction on the construction of the concepts that for every object $x_{\mathcal{R}} \in \Delta^{\mathfrak{R}}$ and every concept C, $x_{\mathcal{R}} \in C^{\mathfrak{R}}$ iff $x \in C^{\mathcal{R}}$.

This, together with the condition that, for every $x_{\mathcal{R}} \in \Delta^{\mathfrak{R}}$, $h_{\mathfrak{R}}(x_{\mathcal{R}}) = h_{\mathcal{R}}(x)$, implies that for every concept C, $h_{\mathcal{R}^{\mathfrak{R}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\}$.

Now, let $C \mathrel{\tilde{\sqsubseteq}} D$ be satisfied by every $\mathcal{R} \in \mathfrak{R}$. Hence, for every $\mathcal{R} \in \mathfrak{R}$, either $h_{\mathcal{R}}(C \sqcap D) < h_{\mathcal{R}}(C \sqcap \neg D)$ or $h_{\mathcal{R}}(C) = \infty$. Since $h_{\mathcal{R}^{\mathfrak{R}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\}$, $h_{\mathcal{R}^{\mathfrak{R}}}(C \sqcap D) = \min\{h_{\mathcal{R}}(C \sqcap D) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\}$, and $h_{\mathcal{R}^{\mathfrak{R}}}(C \sqcap \neg D) = \min\{h_{\mathcal{R}}(C \sqcap \neg D) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\}$, it is easy to check that $\mathcal{R}^{\mathfrak{R}}$ satisfies $C \mathrel{\tilde{\sqsubseteq}} D$ too: Assume that is not the case, that is, $h_{\mathcal{R}^{\mathfrak{R}}}(C \sqcap \neg D) \leq h_{\mathcal{R}^{\mathfrak{R}}}(C \sqcap D)$ and $h_{\mathcal{R}^{\mathfrak{R}}}(C) < \infty$; then, we have that $\min\{h_{\mathcal{R}}(C \sqcap \neg D) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\} \leq \min\{h_{\mathcal{R}}(C \sqcap D) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\}$ and $\min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \in \mathcal{R}^{\mathfrak{R}}\} < \infty$, that, since for every $\mathcal{R} \in \mathfrak{R}$, either $h_{\mathcal{R}}(C \sqcap D) < h_{\mathcal{R}}(C \sqcap \neg D)$ or $h_{\mathcal{R}}(C) = \infty$, cannot be the case. □

Lemma 9. *For every $\mathcal{KB}$ and every $C, D \in \mathcal{L}$, $\mathcal{KB} \models_{\text{mod}} C \mathrel{\tilde{\sqsubseteq}} D$ iff $\mathcal{R} \Vdash C \mathrel{\tilde{\sqsubseteq}} D$, for every $\mathcal{R} \in Mod_{\Delta}(\mathcal{KB})$.*

Proof. Let Δ be a countably infinite domain. For the only-if part, if $\mathcal{KB} \models_{\text{mod}} C \mathrel{\tilde{\sqsubseteq}} D$, then obviously $\mathcal{R} \Vdash C \mathrel{\tilde{\sqsubseteq}} D$ for every $\mathcal{R} \in Mod_{\Delta}(\mathcal{KB})$. For the if part, assume we do not have $\mathcal{KB} \models_{\text{mod}} C \mathrel{\tilde{\sqsubseteq}} D$. Then, thanks to the finite-model property (Theorem 3), there is a modular model $\mathcal{R}_{\text{fin}}$ with a finite domain that is a model of $\mathcal{KB}$ and a counter-model of $C \mathrel{\tilde{\sqsubseteq}} D$; since the domain is finite, the modular model $\mathcal{R}_{\text{fin}}$ is a ranked model (Lemma 7). Given $\mathcal{R}_{\text{fin}}$, we can extend it to a model of $\mathcal{KB}$ that is a counter-model of $C \mathrel{\tilde{\sqsubseteq}} D$ with a countably infinite domain in the following way: make a countably infinite number of copies of $\mathcal{R}_{\text{fin}}$ and make the ranked union of them. Now, let $\mathcal{R}' = \langle \Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}'}, \prec^{\mathcal{R}'} \rangle$ be the result of such ranked union, that is, a ranked model of $\mathcal{KB}$ and a counter-model of $C \mathrel{\tilde{\sqsubseteq}} D$ with $\Delta^{\mathcal{R}'}$ being countably infinite (it is the disjoint union of a countably infinite number of finite domains). It is easy to build an isomorphic interpretation $\mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, once we have defined a bijection $b : \Delta^{\mathcal{R}'} \longrightarrow \Delta$, which must exist, being both $\Delta^{\mathcal{R}'}$ and Δ countably infinite sets. We can define $\cdot^{\mathcal{R}}$ and $\prec^{\mathcal{R}}$ in the following way:

- For every $A \in \mathsf{C}$ and every $x \in \Delta^{\mathcal{R}'}$, $b(x) \in A^{\mathcal{R}}$ iff $x \in A^{\mathcal{R}'}$;
- For every $r \in \mathsf{R}$ and every $x, y \in \Delta^{\mathcal{R}'}$, $(b(x), b(y)) \in r^{\mathcal{R}}$ iff $(x, y) \in r^{\mathcal{R}'}$;
- For every $x \in \Delta^{\mathcal{R}'}$, $h_{\mathcal{R}}(b(x)) = h_{\mathcal{R}'}(x)$.

It is easy to prove by induction on the construction of the concepts that for every $C \in \mathcal{L}$ and every $x \in \Delta^{\mathcal{R}'}$, $x \in C^{\mathcal{R}'}$ iff $b(x) \in C^{\mathcal{R}}$. Moreover, $x \in \min_{\prec^{\mathcal{R}'}}(C^{\mathcal{R}'})$ iff $b(x) \in \min_{\prec^{\mathcal{R}}}(C^{\mathcal{R}})$. Hence, there is a ranked $\mathcal{KB}$-model that is a counter model for $C \mathrel{\tilde{\sqsubseteq}} D$ with Δ as its domain. □

E PROOFS OF RESULTS IN SECTION 5

NB: The results marked (∗) are introduced here in the Appendix, while they are omitted in the main text.

Lemma 10. *For every knowledge base $\mathcal{KB}$ and every concept C, $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$ iff $\mathcal{KB} \models_{\text{mod}} C \sqsubseteq \bot$.*

Proof. If $\mathcal{KB}$ does not have a modular model or C is never satisfiable, then the result is straightforward. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ have a modular model, and let C be satisfiable. Also, let $\mathcal{D}$ be ranked into $\langle \mathcal{D}_0^{\mathsf{rank}}, \ldots, \mathcal{D}_n^{\mathsf{rank}}, \mathcal{D}_\infty^{\mathsf{rank}} \rangle$.

From left to right, let $\text{rank}_{\mathcal{KB}}(C) = \infty$, and assume we do not have $\mathcal{KB} \models_{\text{mod}} C \sqsubseteq \bot$. Together they imply that $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}} \models_{\text{mod}} \top \lesssim \neg C$ but $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}} \models_{\text{mod}} C \sqsubseteq \bot$ does not hold. Hence, due to the FMP (Theorem 3), there is a finite-ranked model $\mathcal{R}$ of $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}}$ with the domain $\Delta^{\mathcal{R}}$ layered into $(L_0^{\mathcal{R}}, \dots, L_n^{\mathcal{R}})$, and s.t. $\mathcal{R} \Vdash \top \lesssim \neg C$ but $\mathcal{R} \Vdash C \sqsubseteq \bot$ does not hold, that is, in $\Delta^{\mathcal{R}}$ there is an object o s.t. $o \in L_i^{\mathcal{R}}$, with $0 < i \leq n$, and $o \in C^{\mathcal{R}}$.

Now let us define a new model $\mathcal{R}'$ simply taking the lower layer and putting it at the "top" of our model, that is, we rearrange the interpretation in the following way:

- $\Delta^{\mathcal{R}'} = \Delta^{\mathcal{R}}$;
- $\cdot^{\mathcal{R}'} = \cdot^{\mathcal{R}}$;
- $L_n^{\mathcal{R}'} = L_0^{\mathcal{R}}$;
- for every $i < n$, $L_i^{\mathcal{R}'} = L_{i+1}^{\mathcal{R}}$.

Clearly for every concept D, $D^{\mathcal{R}'} = D^{\mathcal{R}}$ (it is easy to prove by induction on the construction of the concepts), and consequently $\mathcal{R}'$ is still a model of $\mathcal{T}$. We can prove that is still also a model of $\mathcal{D}_{\infty}^{\text{rank}}$. Assume that is not the case, that is, there is a some $D \lesssim E \in \mathcal{D}_{\infty}^{\text{rank}}$ s.t. $\mathcal{R} \Vdash D \lesssim E$ and not $\mathcal{R}' \Vdash D \lesssim E$. $\mathcal{R} \Vdash D \lesssim E$ if either $h_{\mathcal{R}}(D \sqcap E) < h_{\mathcal{R}}(D \sqcap \neg E)$ or $h_{\mathcal{R}}(D) = \infty$. It cannot be the latter, since $h_{\mathcal{R}}(D) = \infty$ corresponds to $D^{\mathcal{R}} = \emptyset$, and we would have also $D^{\mathcal{R}'} = \emptyset$ and $h_{\mathcal{R}'}(D) = \infty$. Hence, it must be $h_{\mathcal{R}}(D \sqcap E) < h_{\mathcal{R}}(D \sqcap \neg E)$, while $h_{\mathcal{R}}(D \sqcap E) \not< h_{\mathcal{R}}(D \sqcap \neg E)$. Let $h_{\mathcal{R}}(D \sqcap E) = i$ and $h_{\mathcal{R}}(D \sqcap \neg E) = j$ with $i < j$. If $i > 0$, then $h_{\mathcal{R}'}(D \sqcap E) = i - 1$ and $h_{\mathcal{R}'}(D \sqcap \neg E) = j - 1$, and $h_{\mathcal{R}'}(D \sqcap E) < h_{\mathcal{R}'}(D \sqcap \neg E)$ again; hence it must be $h_{\mathcal{R}}(D \sqcap E) = 0$, that is, $h_{\mathcal{R}}(D) = 0$, but that is incompatible with $D \lesssim E \in \mathcal{D}_{\infty}^{\text{rank}}$, since $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}} \models_{\text{mod}} \top \lesssim \neg D$, that is, $h_{\mathcal{R}}(D) > 0$. Consequently, $\mathcal{R}'$ too must be a model of $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}}$.

We have assumed that in $\mathcal{R}$ there is an object o s.t. $o \in C^{\mathcal{R}}$ and $o \in L_i^{\mathcal{R}}$ for some $0 < i \leq n$. Repeating the procedure used to define $\mathcal{R}'$ for i times, we obtain a model $\mathcal{R}^*$ of $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}}$ s.t. $o \in C^{\mathcal{R}^*}$ and $o \in L_0^{\mathcal{R}^*}$. However, since $\text{rank}_{\mathcal{KB}}(C) = \infty$ implies $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}} \models_{\text{mod}} \top \lesssim \neg C$, this cannot be the case. We conclude that if $\text{rank}_{\mathcal{KB}}(C) = \infty$, then $\mathcal{KB} \models_{\text{mod}} C \sqsubseteq \bot$.

From right to left, let $\mathcal{KB} \models_{\text{mod}} C \sqsubseteq \bot$ but $\text{rank}_{\mathcal{KB}}(C) \neq \infty$. The latter implies that there is a model of $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}}$ that does not satisfy $\top \lesssim \neg C$, that is, does not satisfy $C \sqsubseteq \bot$. Referring again to the FMP (Theorem 3), we can say that there is a finite-ranked model $\mathcal{R}$ of $\mathcal{T} \cup \mathcal{D}_{\infty}^{\text{rank}}$ that does not satisfy $C \sqsubseteq \bot$. Let k be the number of layers in $\mathcal{R}$.

Now consider $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_{\infty}^{\text{rank}})$. For each $D \lesssim E \in \mathcal{D}_n^{\text{rank}}$ there must be a model in which $D \sqcap E$ is not exceptional, that is, it is satisfied in the layer 0. As a consequence, still using the FMP (Corollary 6), for each $D \lesssim E \in \mathcal{D}_n^{\text{rank}}$ there must a finite-ranked model $\mathcal{R}_{D \lesssim E}$ of $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_{\infty}^{\text{rank}})$ s.t. $h_{\mathcal{R}_{D \lesssim E}}(D \lesssim E) = 0$.

Build a ranked interpretation $\mathcal{R}_n$ as follows:

- for every $D \lesssim E \in \mathcal{D}_n^{\text{rank}}$, let $\mathcal{R}_{D \lesssim E}$ be a finite-ranked model of $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_{\infty}^{\text{rank}})$ in which $h_{\mathcal{R}_{D \lesssim E}}(D \lesssim E) = 0$.
- Let $\mathcal{R}' = \langle \Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}'}, \prec^{\mathcal{R}'} \rangle$ be the ranked union of such sets. $\mathcal{R}'$ is a model of $\mathcal{D}_n^{\text{rank}}$ (Lemma 8) s.t. for every $D \lesssim E \in \mathcal{D}_n^{\text{rank}}$, $h_{\mathcal{R}_{D \lesssim E}}(D \lesssim E) = 0$. Since $\mathcal{D}_n^{\text{rank}}$ is finite, it has been obtained from a finite set of finite models and so it is a finite-ranked model. Let m be the number of layers in $\mathcal{R}'$.
- From $\mathcal{R}'$ and $\mathcal{R}$ define a finite-ranked interpretation $\mathcal{R}_n = \langle \Delta^{\mathcal{R}_n}, \cdot^{\mathcal{R}_n}, \prec^{\mathcal{R}_n} \rangle$ as follows:
  - $\Delta^{\mathcal{R}_n} = \Delta^{\mathcal{R}} \cup \Delta^{\mathcal{R}'}$;
  - $A^{\mathcal{R}_n} = A^{\mathcal{R}} \cup A^{\mathcal{R}'}$ for every $A \in \mathsf{C}$;

- $r^{\mathcal{R}_n} = r^{\mathcal{R}} \cup r^{\mathcal{R}'}$ for every $r \in \mathsf{R}$;
- for every $i \leq m$, $L_i^{\mathcal{R}_n} = L_i^{\mathcal{R}'}$;
- for every $m < i \leq (m + k)$, $L_i^{\mathcal{R}_n} = L_{(i-(m+1))}^{\mathcal{R}}$.

Informally, we build the model $\mathcal{R}_n$ by adding $\mathcal{R}$ on top of $\mathcal{R}'$. It is easy to prove by induction on the construction of the concepts that every object in $\mathcal{R}_n$ satisfies a concept D iff it satisfies D also in the original model, $\mathcal{R}$ or $\mathcal{R}'$. As a consequence, we do not have $\mathcal{R}_n \Vdash C \sqsubseteq \bot$. Also, it is easy to prove that $\mathcal{R}_n$ is a model of $\mathcal{T} \cup (\mathcal{D}_n^{\mathsf{rank}} \cup \mathcal{D}_\infty^{\mathsf{rank}})$: $\mathcal{R}'$ is a model of $\mathcal{D}_n^{\mathsf{rank}}$ with at layer 0 an object satisfying $D \sqcap E$ for each $D \precsim E \in \mathcal{D}_n^{\mathsf{rank}}$, and both $\mathcal{R}$ and $\mathcal{R}'$ are models of $\mathcal{T} \cup \mathcal{D}_\infty^{\mathsf{rank}}$.

Now consider $\mathcal{T} \cup (\mathcal{D}_{(n-1)}^{\mathsf{rank}} \cup \mathcal{D}_n^{\mathsf{rank}} \cup \mathcal{D}_\infty^{\mathsf{rank}})$. Using the same procedure defined for $\mathcal{R}_n$, we can build a model $\mathcal{R}_{n-1}$, obtained doing the ranked union of a finite set of finite models of $\mathcal{T} \cup (\mathcal{D}_{(n-1)}^{\mathsf{rank}} \cup \mathcal{D}_n^{\mathsf{rank}} \cup \mathcal{D}_\infty^{\mathsf{rank}})$ and adding on top $\mathcal{R}_n$. $\mathcal{R}_{n-1}$ will be a finite-ranked model of $\mathcal{T} \cup (\mathcal{D}_{(n-1)}^{\mathsf{rank}} \cup \mathcal{D}_n^{\mathsf{rank}} \cup \mathcal{D}_\infty^{\mathsf{rank}})$ s.t. $\mathcal{R}_{n-1} \Vdash C \sqsubseteq \bot$ does not hold.

We can go on with this procedure until we define a finite-ranked model $\mathcal{R}_0$ of $\mathcal{T} \cup (\mathcal{D}_0^{\mathsf{rank}} \cup \ldots \cup \mathcal{D}_n^{\mathsf{rank}} \cup \mathcal{D}_\infty^{\mathsf{rank}})$. That is, $\mathcal{R}_0$ is a model of $\mathcal{T} \cup \mathcal{D}$ s.t. we do not have $\mathcal{R}_0 \Vdash C \sqsubseteq \bot$, against the hypothesis that $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq \bot$. □

To prove Theorem 4, we will use the following lemma.

Lemma 43 (*). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be a defeasible knowledge base having a modular model, $\mathcal{O}$ its big ranked model, and Δ the countably infinite domain used to define $\mathcal{O}$. For every $C \precsim D \in \mathcal{D}$,* $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = i$ *iff there is a model $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $h_{\mathcal{R}_\Delta}(C \sqcap D) = i$.*

Proof. First, we observe that the exceptionality function in Definition 23 is correctly captured in the model $\mathcal{O}$, that is, for every $C \in \mathcal{L}$, $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim \neg C$ iff $\mathcal{O} \Vdash \top \precsim \neg C$. Indeed, by Lemma 9, a concept C is exceptional w.r.t. $\mathcal{KB}$ iff $\mathcal{R}_\Delta \Vdash \top \precsim \neg C$, for every $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$, which immediately corresponds to $\mathcal{O} \Vdash \top \precsim \neg C$.

Since $\mathcal{R}_\Delta \Vdash \mathcal{KB}$ for every $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$, if $h_{\mathcal{R}_\Delta}(C) = i$, it is immediate that $\mathsf{rank}_{\mathcal{KB}}(C) \leq i$, otherwise it would be $h_{\mathcal{R}_\Delta}(C) > i$ for every $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$. We have to prove that for every $C \precsim D \in \mathcal{D}$, if $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = i$, then there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $h_{\mathcal{R}_\Delta}(C \sqcap D) = i$. In case $i = \infty$, Lemma 10 guarantees that if $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = \infty$, then for all the $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$, $h_{\mathcal{R}_\Delta}(C \sqcap D) = \infty$. In case $i < \infty$, we can prove it by induction on the ranking value i.

Let $C \precsim D \in \mathcal{D}$, and let $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = i$. For $i = 0$, we already have all that is needed to prove that there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $\mathcal{R}_\Delta \Vdash \top \precsim \neg(C \sqcap D)$ is not the case:

- $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = 0$, iff we do not have $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim \neg(C \sqcap D)$ (Definition 23);
- $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim \neg(C \sqcap D)$ does not hold iff there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $\mathcal{R}_\Delta \Vdash \top \precsim \neg(C \sqcap D)$ is not the case (by Lemma 9);
- We do not have $\mathcal{R}_\Delta \Vdash \top \precsim \neg(C \sqcap D)$ iff $h_{\mathcal{R}_\Delta}(C \sqcap D) = 0$.

For $i > 0$, we can define a modular model $\mathcal{R}$ of $\mathcal{KB}$ as follows:

Let $C \precsim D \in \mathcal{D}$ with $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = i$, and let $\mathcal{D}_{\geq i}^{\mathsf{rank}}$ be the subset of $\mathcal{D}$ containing the DCIs with a ranking value of at least i, and $\mathcal{D}_{<i}^{\mathsf{rank}} = \mathcal{D} \setminus \mathcal{D}_{\geq i}^{\mathsf{rank}}$. Let $\mathcal{R}'$ be a modular model of $\mathcal{T} \cup \mathcal{D}_{\geq i}^{\mathsf{rank}}$ such that $h_{\mathcal{R}'}(C \sqcap D) = 0$. Such a model must exist, since $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = i$, that is, $C \sqcap D$ is not exceptional in $\mathcal{T} \cup \mathcal{D}_{\geq i}^{\mathsf{rank}}$. We can assume that $\mathcal{R}'$ has a finite domain, given the finite-model property (Corollary 6), and hence it is a ranked model (Lemma 7).

For each DCI $D \precsim E \in \mathcal{D}_{<i}^{\mathsf{rank}}$, that is, such that $\mathsf{rank}_{\mathcal{KB}}(D \sqcap E) = j$ for some $j < i$, let $\mathcal{R}_{D \sqcap E} \in Mod_\Delta(\mathcal{KB})$ be a model of $\mathcal{KB}$ satisfiying $D \sqcap E$ s.t. $h_{\mathcal{R}_{D \sqcap E}}(D \sqcap E) = j$. The induction hypothesis guarantees that such a model exists for each such DCI.

Now, we define a new interpretation $\mathcal{R}'' = \langle \Delta^{\mathcal{R}''}, \cdot^{\mathcal{R}''}, \prec^{\mathcal{R}''} \rangle$ in the following way:

- $\Delta^{\mathcal{R}''} = \Delta^{\mathcal{R}'} \cup \bigcup_{(C \sqsubsim D \in \mathcal{D}^{\mathsf{rank}}_{<i})} \Delta^{\mathcal{R}_{C \sqcap D}}$;
- For every concept name $A \in \mathsf{C}$ and every $x \in \Delta^{\mathcal{R}''}$, $x \in A^{\mathcal{R}''}$ iff one of the two following cases holds: either $x \in \Delta^{\mathcal{R}_{D \sqcap E}}$ for some $D \sqsubsim E \in \mathcal{D}^{\mathsf{rank}}_{<i}$ and $x \in A^{\mathcal{R}_{C \sqcap D}}$, or $x \in \Delta^{\mathcal{R}'}$ and $x \in A^{\mathcal{R}'}$;
- For every role name $r \in \mathsf{R}$ and every $x, y \in \Delta^{\mathcal{R}''}$, $(x, y) \in r^{\mathcal{R}''}$ iff one of the two following cases holds: either $x, y \in \Delta^{\mathcal{R}_{C \sqcap D}}$ for some $C \sqsubsim D \in \mathcal{D}^{\mathsf{rank}}_{<i}$ and $(x, y) \in r^{\mathcal{R}''}$, or $x, y \in \Delta^{\mathcal{R}'}$ and $(x, y) \in r^{\mathcal{R}'}$;
- For every $x \in \Delta^{\mathcal{R}''}$, $h_{\mathcal{R}''}(x) = j$ iff one of the two following cases holds: either $x \in \Delta^{\mathcal{R}_{D \sqcap E}}$ for some $D \sqsubsim E \in \mathcal{D}^{\mathsf{rank}}_{<i}$ and and $h_{\mathcal{R}_{D \sqcap E}}(x) = j$, or $x \in \Delta^{\mathcal{R}'}$ and $h_{\mathcal{R}'}(x) = j - i$.

The idea is to create a model of $\mathcal{KB}$ that guarantees for a specific inclusion $C \sqsubsim D \in \mathcal{D}$ that the height of C in the model corresponds exactly to the rank of C. That is, given an inclusion $C \sqsubsim D$ that has rank i, we have built a ranked interpretation $\mathcal{R}''$ in which C has height i. Now, we need to:

- Prove that $\mathcal{R}''$ is a model of $\mathcal{KB}$;
- Show that an isomorphic model to $\mathcal{R}''$ is in $Mod_\Delta(\mathcal{KB})$.

It can easily be proven that $\mathcal{R}''$ is a model of $\mathcal{KB}$: First, we prove by induction on the construction of concepts that, for every $x \in \Delta^{\mathcal{R}''}$, $x \in D^{\mathcal{R}''}$ iff the corresponding object falls under D in the original model; this grants us that $\mathcal{R}''$ satisfies $\mathcal{T}$. About the satisfaction of $\mathcal{D}$, referring to the height values that have been assigned to each object in $\mathcal{R}''$, we can prove that for every $D \sqsubsim E \in \mathcal{D}$, $h_{\mathcal{R}''}(D \sqcap E) < h_{\mathcal{R}''}(D \sqcap \neg E)$ (or $h_{\mathcal{R}''}(D) = \infty$). Hence, $\mathcal{R}''$ is a model of $\mathcal{KB}$. Also, notice that in $\mathcal{R}'$, we must have an object o s.t. $h_{\mathcal{R}'}(o) = 0$ and $o \in (C \sqcap D)^{\mathcal{R}'}$. The construction of of $\mathcal{R}''$ implies that $h_{\mathcal{R}'}(o) = i$ and $o \in (C \sqcap D)^{\mathcal{R}''}$. That is, $\mathcal{R}''$ is a model of $\mathcal{KB}$ in which $h_{\mathcal{R}''}(C \sqcap D) = i$.

$\Delta^{\mathcal{R}''}$ has been created unifying a finite number of model with the countably infinite domain Δ plus the finite domain $\Delta^{\mathcal{R}'}$, hence $\Delta^{\mathcal{R}''}$ has a countably infinite domain, and there is a model $\mathcal{R}''_\Delta$ that is isomorphic to $\mathcal{R}''$ and has Δ as domain. □

Using Lemma 43, we can prove Theorem 4.

Theorem 4. *Let $\mathcal{KB}$ be a defeasible knowledge base having a modular model. A statement α is in the rational closure of $\mathcal{KB}$ iff $\mathcal{KB} \models_{\mathsf{rat}} \alpha$.*

Proof. Let $\mathcal{KB}$ be a defeasible knowledge base with a modular model, $\mathcal{O}$ the big ranked model of $\mathcal{KB}$, and Δ the countably infinite domain used to define $\mathcal{O}$. $\mathcal{KB} \models_{\mathsf{rat}} \alpha$ iff $\mathcal{O} \Vdash \alpha$ (Definition 22), so we need to prove that $\mathcal{O} \Vdash \alpha$ iff α is in the rational closure of $\mathcal{KB}$. We first prove the result where α is a DCI (of the form $C \sqsubsim D$), that is, we need to prove that $\mathcal{O} \Vdash C \sqsubsim D$ iff $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) < \mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D)$ or $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$. In turn, that means that we need to prove that for every DCI $C \sqsubsim D$, $h_{\mathcal{O}}(C \sqcap D) < h_{\mathcal{O}}(C \sqcap \neg D)$ or $h_{\mathcal{O}}(C) = \infty$ iff $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) < \mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D)$ or $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$. Such a result follows immediately if we can prove that, for every concept C, $h_{\mathcal{O}}(C) = \mathsf{rank}_{\mathcal{KB}}(C)$, and that is what we are going to do.

An immediate consequence of Lemma 43 is that, for every $C \sqsubsim D \in \mathcal{D}$, $h_{\mathcal{O}}(C \sqcap D) = \mathsf{rank}_{\mathcal{KB}}(C \sqcap D)$. Being $\mathcal{O}$ a model of $\mathcal{D}$, if $C \sqsubsim D \in \mathcal{D}$ then $h_{\mathcal{O}}(C \sqcap D) = h_{\mathcal{O}}(C)$ and $\mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = \mathsf{rank}_{\mathcal{KB}}(C)$. So, for every $C \sqsubsim D \in \mathcal{D}$, $h_{\mathcal{O}}(C) = h_{\mathcal{O}}(C \sqcap D) = \mathsf{rank}_{\mathcal{KB}}(C \sqcap D) = \mathsf{rank}_{\mathcal{KB}}(C)$. Now, we extend such a result to any concept C, using a construction that is in line with the one used to prove Lemma 43.

Since $\mathcal{O} \Vdash \mathcal{KB}$, if $h_{\mathcal{O}}(C) = i$, it is immediate that $\mathsf{rank}_{\mathcal{KB}}(C) \leq i$, otherwise it would be $h_{\mathcal{O}}(C) > i$. We have to prove that for every concept C, if $\mathsf{rank}_{\mathcal{KB}}(C) = i$, then $h_{\mathcal{O}_\Delta}(C) = i$, that is, there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $h_{\mathcal{R}_\Delta}(C) = i$. In case $i = \infty$, Lemma 10 guarantees that if $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$, then for all the $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$, $h_{\mathcal{R}_\Delta}(C) = \infty$. In case $i < \infty$, we can prove it by induction on the ranking value i.

Let $\mathsf{rank}_{\mathcal{KB}}(C) = i$. For $i = 0$, we already have all that is needed to prove that there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $\mathcal{R}_\Delta \Vdash \top \precsim \neg(C)$ is not the case:

- $\mathsf{rank}_{\mathcal{KB}}(C) = 0$, iff we do not have $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim \neg(C)$ (Definition 23);
- We do not have $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim \neg(C)$ iff there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $\mathcal{R}_\Delta \Vdash \top \precsim \neg(C)$ does not hold (by Lemma 9);
- It is not the case that $\mathcal{R}_\Delta \Vdash \top \precsim \neg(C)$ iff $h_{\mathcal{R}_\Delta}(C) = 0$.
- $h_{\mathcal{O}}(C) = 0$ iff there is a $\mathcal{R}_\Delta \in Mod_\Delta(\mathcal{KB})$ s.t. $h_{\mathcal{R}_\Delta}(C) = 0$.

For $i > 0$, we can define a modular model $\mathcal{R}$ of $\mathcal{KB}$ as follows:

Let $\mathsf{rank}_{\mathcal{KB}}(C) = i$, and, as in Lemma 43, let $\mathcal{D}^{\mathsf{rank}}_{\geq i}$ be the subset of $\mathcal{D}$ containing the DCIs with a ranking value of at least i, and $\mathcal{D}^{\mathsf{rank}}_{<i} = \mathcal{D} \setminus \mathcal{D}^{\mathsf{rank}}_{\geq i}$. Let $\mathcal{R}'$ be a modular model of $\mathcal{T} \cup \mathcal{D}^{\mathsf{rank}}_{\geq i}$ such that $h_{\mathcal{R}'}(C) = 0$. Such a model must exist, since $\mathsf{rank}_{\mathcal{KB}}(C) = i$, that is, C is not exceptional in $\mathcal{T} \cup \mathcal{D}^{\mathsf{rank}}_{\geq i}$. We can assume that $\mathcal{R}'$ has a finite domain, given the finite-model property (Corollary 6), and hence it is a ranked model (Lemma 7).

Then, we define an interpretation $\mathcal{R}''$ exactly as done in the proof of Lemma 43, and, exactly as in Lemma 43, we can prove that $\mathcal{R}''$ is a model of $\mathcal{KB}$ with a countably infinite domain and s.t. $h_{\mathcal{R}''}(C) = i$.

That implies that there is a model $\mathcal{R}''_\Delta \in Mod_\Delta(\mathcal{KB})$ that is isomorphic to $\mathcal{R}''$, with $h_{\mathcal{R}''_\Delta}(C) = i$.

Since $\mathcal{R}''_\Delta$ is used in the construction of $\mathcal{O}$, $h_{\mathcal{O}}(C) \leq i$; since $\mathsf{rank}_{\mathcal{KB}}(C) = i$, $h_{\mathcal{O}}(C) \geq i$. Hence, $h_{\mathcal{O}}(C) = i$.

For the case where α is a GCI (of the form $C \sqsubseteq D$), suppose that $C \sqsubseteq D$ is in the rational closure of $\mathcal{KB}$. By definition this means that $\mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D) = \infty$. By Lemma 10, we then have that $\mathcal{KB} \models_{\mathsf{mod}} C \sqcap \neg D \sqsubseteq \bot$, which means that $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq D$. And, since $\mathcal{O}$ is a ranked model of $\mathcal{KB}$, it follows that $\mathcal{KB} \models_{\mathsf{rat}} C \sqsubseteq D$. Conversely, suppose that $\mathcal{O}$ is a ranked model of $C \sqsubseteq D$. Then $\mathcal{O}$ is a ranked model of $C \sqcap \neg D \precsim \bot$, from which it follows, by Lemma 9, that $\mathcal{KB} \models_{\mathsf{mod}} C \sqcap \neg D \precsim \bot$. It then follows from Lemma 10 that $\mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D) = \infty$. And this mean, by definition, that $C \sqsubseteq D$ is in the rational closure of $\mathcal{KB}$. ◻

Lemma 11. *For $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, if $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C$, then $C \precsim D$ is exceptional w.r.t. $\mathcal{T} \cup \mathcal{D}$.*

Proof. It suffices to prove that if we do not have $\mathcal{T} \cup \mathcal{D} \models_{\mathsf{mod}} \top \precsim \neg C$, then $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C$ does not hold. So, suppose that $\mathcal{T} \cup \mathcal{D} \models_{\mathsf{mod}} \top \precsim \neg C$ is not the case. This means there is a modular model $\mathcal{R}$ of $\mathcal{T} \cup \mathcal{D}$ for which we have an $x \in \Delta^{\mathcal{R}}$ such that $x \in C^{\mathcal{R}}$. Let $\mathcal{I}$ be the classical interpretation associated with $\mathcal{R}$. It follows immediately that $\mathcal{I}$ is a model of $\mathcal{T}$ and that $x \in (\bigsqcap \overline{\mathcal{D}})^{\mathcal{I}}$, but that $x \notin (\neg C)^{\mathcal{I}}$. ◻

Lemma 44 (∗). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$. Then* (i) *$\mathcal{KB} \subseteq Cn_{\mathsf{rat}}(\mathcal{KB})$ and* (ii) *$Cn_{\mathsf{rat}}(\mathcal{KB})$ induces a defeasible subsumption relation $\precsim^{\mathcal{KB}}_{\mathsf{rat}} =_{\mathsf{def}} \{(C, D) \mid \mathcal{KB} \vdash_{\mathsf{rat}} C \precsim D\}$ that is rational.*

Proof. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$.

Proving (*i*): Assume $C \sqsubseteq D \in \mathcal{T}$. $\mathcal{KB} \vdash_{\mathsf{rat}} C \sqsubseteq D$ iff $\mathcal{T}^* \models C \sqsubseteq D$; since $\mathcal{T} \subseteq \mathcal{T}^*$, $\mathcal{T} \subseteq Cn_{\mathsf{rat}}(\mathcal{KB})$. Assume that $C \precsim D \in \mathcal{D}$. Either $C \precsim D$ ends up in $\mathcal{D}^*_\infty$, or there will be an i ($0 \leq i \leq n$) s.t. $\mathsf{rk}(C) = \mathsf{rk}(C \precsim D) = i$. In the former case, $C \sqsubseteq D$ is in $\mathcal{T}^*$, and so $\mathcal{T}^* \models C \sqsubseteq D$, i.e., $\mathcal{KB} \vdash_{\mathsf{rat}} C \precsim D$. In the latter case, $\models \overline{\mathcal{E}_i} \sqsubseteq \neg C \sqcup D$, and so $\mathcal{T}^* \models \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$, i.e., $C \precsim D \in Cn_{\mathsf{rat}}(\mathcal{KB})$. Hence, $\mathcal{T} \cup \mathcal{D} \subseteq Cn_{\mathsf{rat}}(\mathcal{KB})$.

Proving (*ii*): Let $\precsim^{\mathcal{KB}}_{\mathsf{rat}} =_{\mathsf{def}} \{(C, D) \mid \mathcal{KB} \vdash_{\mathsf{rat}} C \precsim D\}$. We show $\precsim^{\mathcal{KB}}_{\mathsf{rat}}$ satisfies all rationality properties.

- (Ref). Since $\models C \sqsubseteq C$ is valid for any $C \in \mathcal{L}$, we have that $\mathcal{T}^* \models \overline{\mathcal{E}_i} \sqcap C \sqsubseteq C$ for any $\mathcal{T}^*$ and $\overline{\mathcal{E}_i}$.
- (LLE). $C \precsim E \in Cn_{\mathsf{rat}}(\mathcal{KB})$ implies that $\mathcal{T}^* \models \overline{\mathcal{E}_i} \sqcap C \sqsubseteq E$ for some i (or $\mathcal{T}^* \models C \sqsubseteq E$, if $\mathsf{rk}(C) = \infty$). Since $\models C \equiv D$, $\overline{\mathcal{E}_i}$ is the lowest i s.t. we do not have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg D$, and $\mathcal{T}^* \models \overline{\mathcal{E}_i} \sqcap D \sqsubseteq E$, too.
- (And). $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$ and $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq E$ (possibly without $\bigsqcap \overline{\mathcal{E}_i}$, if C has an infinite rank), hence $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D \sqcap E$, that is, $C \precsim D \sqcap E \in Cn_{\mathsf{rat}}(\mathcal{KB})$.
- (Or). $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq E$ for some i and $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_j} \sqcap D \sqsubseteq E$ for some j. Assume that $i \leq j$ and $i < \infty$, that is, $\models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \bigsqcap \overline{\mathcal{E}_j}$. Then, since we do not have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg C$, we do not have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg(C \sqcup D)$ either. Moreover $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_j} \sqcap D \sqsubseteq E$ and $\models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \bigsqcap \overline{\mathcal{E}_j}$ imply that $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap D \sqsubseteq E$. So, $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap (C \sqcup D) \sqsubseteq E$. The proof is analogous for $j \leq i$ with $j < \infty$, or if i and j correspond to ∞.
- (RW). $C \precsim D \in Cn_{\mathsf{rat}}(\mathcal{KB})$ if $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$ for some $\bigsqcap \overline{\mathcal{E}_i}$ (or $\mathcal{T}^* \models C \sqsubseteq D$, if $\mathsf{rk}(C) = \infty$). Since $\models D \sqsubseteq E$, $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq E$.
- (CM). If $\mathsf{rk}(C) = i < \infty$, then $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$ and $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq E$ for some $\bigsqcap \overline{\mathcal{E}_i}$. Since $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$ and we do not have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg C$, $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg(C \sqcap D)$ does not hold; otherwise, we would have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D \sqcap \neg D$, i.e., $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg C$. Hence, we have $C \sqcap D \precsim E \in Cn_{\mathsf{rat}}(\mathcal{KB})$, since $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqcap D \sqsubseteq E$. If $\mathsf{rk}(C) = \infty$, then we have $\mathcal{T}^* \models C \sqsubseteq \bot$, and the proof is trivial.
- (RM). If $\mathsf{rk}(C) = i < \infty$, then $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq E$ and $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \neg D$ does not hold for some $\bigsqcap \overline{\mathcal{E}_i}$. Since we do not have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \neg D$, $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg(C \sqcap D)$ does not hold; otherwise, we would have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \neg D$. Hence, we have $C \sqcap D \precsim E \in Cn_{\mathsf{rat}}(\mathcal{KB})$, since $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqcap D \sqsubseteq E$. If $\mathsf{rk}(C) = \infty$, then we have $\mathcal{T}^* \models C \sqsubseteq \bot$, and the proof is trivial. □

The following lemma states that, as in the propositional case [70], our procedure correctly manages the classical information, that is, an axiom $C \precsim \bot$ is in the rational closure of $\mathcal{KB}$ if and only if it is also a modular consequence of $\mathcal{KB}$.

Lemma 45 (∗). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and assume $C \precsim D \in \mathcal{D}$. Then $\mathcal{KB} \models_{\mathsf{mod}} C \precsim \bot$ iff $\mathsf{rk}(C) = \infty$ iff $\mathcal{T}^* \models C \sqsubseteq \bot$.*

Proof. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$.

For the only-if part, $\mathcal{KB} \models_{\mathsf{mod}} C \precsim \bot$ implies that every rational subsumption relation containing $\mathcal{KB}$ must satisfy also $C \precsim \bot$. Hence, we have that $\mathcal{KB} \vdash_{\mathsf{rat}} C \precsim \bot$, since $Cn_{\mathsf{rat}}(\mathcal{KB})$ induces a rational subsumption relation extending $\mathcal{KB}$ (Lemma 44). From Definition 27, we know that $\mathcal{KB} \vdash_{\mathsf{rat}} C \precsim \bot$ is possible only if C is always negated in the ranking procedure, i.e., $\mathcal{T}^* \models C \sqsubseteq \bot$.

For the if part, we define from $\mathcal{KB}$ a new knowledge base $\mathcal{KB}^* =_{\mathsf{def}} \mathcal{T}^* \cup \mathcal{D}^*$, with $\mathcal{T}^*$ obtained from $\mathcal{T}$ by adding all the sets $\{C \sqsubseteq D \mid C \precsim D \in \mathcal{D}^*_\infty\}$ that we obtain at each iteration of function ComputeRanking(·). Let us denote with $\mathcal{D}^1_\sqsubseteq, \ldots, \mathcal{D}^n_\sqsubseteq$ such sets. Assume that $\mathcal{T}^* \models C \sqsubseteq \bot$, but that we do not have $\mathcal{KB} \models_{\mathsf{mod}} C \precsim \bot$, i.e., there is a modular model of $\mathcal{KB}$ in which C is non-empty. Let $\mathcal{R}$ be such a model, with an object x falling under $C^\mathcal{R}$. Since $\mathcal{T}^* \models C \sqsubseteq \bot$, there must be a GCI $E \sqsubseteq F$ in some $\mathcal{D}^i_\sqsubseteq$ that is not satisfied, that is, given the nature of the GCIs in every $\mathcal{D}^n_\sqsubseteq$ ($\mathcal{T}^* \models E \sqsubseteq \bot$ for every $E \sqsubseteq F$ contained in some $\mathcal{D}^n_\sqsubseteq$), this means that there is a subsumption $E \sqsubseteq \bot$ that is not satisfied in $\mathcal{R}$. Therefore, there must be an object y falling under $E^\mathcal{R}$. Hence, assuming $E \sqsubseteq F \in \mathcal{D}^i_\sqsubseteq$, since $\mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq \models \bigsqcap\{\neg G \sqcup H \mid G \sqsubseteq H \in \mathcal{D}^i_\sqsubseteq\} \sqsubseteq \neg E$, either $\mathcal{R} \Vdash$

$\mathcal{T} \cup \mathcal{D}^1_{\sqsubseteq} \cup \ldots \cup \mathcal{D}^{i-1}_{\sqsubseteq}$ and $y \in (G \sqcap \neg H)^{\mathcal{R}}$ for some $G \sqsubseteq H \in \mathcal{D}^i_{\sqsubseteq}$ (Case 1 below), or we do not have $\mathcal{R} \Vdash \mathcal{T} \cup \mathcal{D}^1_{\sqsubseteq} \cup \ldots \cup \mathcal{D}^{i-1}_{\sqsubseteq}$ (Case 2 below).

Case 1. Since $\mathcal{R} \Vdash \mathcal{KB}$, $\mathcal{R}$ is also a model of $G \precsim H$, which is an element of $\mathcal{D}$. Hence, there must be an object y such that $y \prec^{\mathcal{R}} x$ (remember that $x \in C^{\mathcal{R}}$) and $y \in (G \sqcap H)^{\mathcal{R}}$. Again, since $G \sqsubseteq H \in \mathcal{D}^i_{\sqsubseteq}$ (which implies $\mathcal{T} \cup \mathcal{D}^1_{\sqsubseteq} \cup \ldots \cup \mathcal{D}^{i-1}_{\sqsubseteq} \models \bigsqcap\{\neg G \sqcup H \mid G \sqsubseteq H \in \mathcal{D}^i_{\sqsubseteq}\} \sqsubseteq \neg G$) and $\mathcal{R} \Vdash \mathcal{T} \cup \mathcal{D}^1_{\sqsubseteq} \cup \ldots \cup \mathcal{D}^{i-1}_{\sqsubseteq}$, there must be a GCI $I \sqsubseteq L \in \mathcal{D}^i_{\sqsubseteq}$ such that $y \in (I \sqcap \neg L)^{\mathcal{R}}$, and we need an object z such that $z \prec^{\mathcal{R}} y$ and $z \in (H \sqcap I)^{\mathcal{R}}$, and so on...This procedure creates an infinitely descending chain of objects, and, since the number of the antecedents of the axioms in $\mathcal{D}^i_{\infty}$ is finite, it cannot be the case, since the model would not satisfy the smoothness condition for the concept $\bigsqcup\{C \mid C \precsim D \in \mathcal{D}^i_{\infty}\}$ (see Definition 5).

Case 2. If it is not the case that $\mathcal{R} \Vdash \mathcal{T} \cup \mathcal{D}^1_{\sqsubseteq} \cup \ldots \cup \mathcal{D}^{i-1}_{\sqsubseteq}$, then $\mathcal{R}$ does not satisfy some $E \sqsubseteq F \in \mathcal{D}^j_{\sqsubseteq}$ for some $j < i$, and therefore there must be an object falling under $E^{\mathcal{R}}$. Again, it is either Case 1 or Case 2. Nevertheless, since at every iteration of Case 2 we pick a lower value j for $\mathcal{D}^j_{\sqsubseteq}$ and we have a finite sequence of $\mathcal{D}^j_{\sqsubseteq}$, we know that after some steps (in the worst case, when we reach $\mathcal{D}^0_{\sqsubseteq}$), we necessarily fall into Case 1, which cannot be the case. □

An immediate consequence of Lemma 45 binds preferential consistency (existence of a preferential model—cf. Definition 6) to classical consistency.

Corollary 7 (∗). *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$. Then $\mathcal{KB} \models_{\text{mod}} \top \precsim \bot$ iff $\mathcal{T}^* \models \top \sqsubseteq \bot$.*

We can now prove that the knowledge bases $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ (in rank normal form) are modularly equivalent.

Lemma 12. *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and let $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ be obtained from $\mathcal{KB}$ through function* ComputeRanking(·). *Then $\mathcal{KB}$ and $\mathcal{KB}^*$ are modularly equivalent.*

Proof. Given $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, the function ComputeRanking($\mathcal{KB}$) outputs a knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$, in which the iteration of lines 5–13 identifies a (possibly empty) set $\{C_1 \precsim D_1, \ldots, C_n \precsim D_n\}$ of always exceptional defeasible subsumptions, that is moved from $\mathcal{D}$ to $\mathcal{T}$. That is, we have $\mathcal{T}^* = \mathcal{T} \cup \{C_1 \sqsubseteq D_1, \ldots, C_n \sqsubseteq D_n\}$ and $\mathcal{D}^* = \mathcal{D} \setminus \{C_1 \precsim D_1, \ldots, C_n \precsim D_n\}$. It is sufficient to prove that $\mathcal{KB} \models_{\text{mod}} C_i \sqsubseteq \bot$ and $\mathcal{KB}^* \models_{\text{mod}} C_i \precsim D_i$ for every $C_i \precsim D_i$ ($1 \leq i \leq n$).

Let $C_i \precsim D_i \in \mathcal{D} \setminus \mathcal{D}^*$. It means that, at some iteration through Lines 4–14 of function ComputeRanking(·), we have $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{D}^*_{\infty}} \sqsubseteq \neg C_i$, which implies that $\mathcal{T}^* \cup \mathcal{D}^{*\sqsubseteq}_{\infty} \models \top \sqsubseteq \neg C_i$, where $\mathcal{D}^{*\sqsubseteq}_{\infty} =_{\text{def}} \{C \sqsubseteq D \mid C \precsim D \in \mathcal{D}^*_{\infty}\}$). Since every $\mathcal{D}^{*\sqsubseteq}_{\infty}$ created at every iteration is contained in the final $\mathcal{T}^*$, using such final $\mathcal{T}^*$, we have that $\mathcal{T}^* \models C_i \sqsubseteq \bot$. Hence, by Lemma 45, we have that $\mathcal{KB} \models_{\text{mod}} C_i \precsim \bot$, i.e., $\mathcal{KB} \models_{\text{mod}} C_i \sqsubseteq \bot$.

However, if $C_i \precsim D_i \in \mathcal{D} \setminus \mathcal{D}^*$, then $C_i \sqsubseteq D_i \in \mathcal{T}^*$, and hence $\mathcal{KB}^* \models_{\text{mod}} C_i \precsim D_i$ by supraclassicality (cf. proof of Lemma 46 below). □

Now, we are justified in using the rank normal form $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ to analyse the rational closure of the knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$. Hence, in what follows, we shall assume that the knowledge bases we are working with are already in rank normal form (and therefore $\mathcal{D}_{\infty} = \emptyset$).

In the next lemma, we observe that the inference relation $\vdash_{\text{rat}}$ respects the preferential conclusions of $\mathcal{KB}$ w.r.t. assertions of the form $\top \precsim C$, another desideratum proven for the propositional case by Lehmann and Magidor [70].

Lemma 46 (∗). *For every $C \in \mathcal{L}$, $\mathcal{KB} \models_{\text{mod}} \top \precsim C$ iff $\mathcal{KB} \vdash_{\text{rat}} \top \precsim C$.*

Proof. First, recall that $\mathcal{KB} \vdash_{\text{rat}} \top \precsim C$ if $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{D}^*} \sqsubseteq C$ (cf. Definition 27).

For the if part, first, we need to prove two properties of $\models_{mod}$, namely, supra-classicality (Sup) and one half of the deduction theorem (S):

$$(\mathrm{Sup})\ \frac{C \sqsubseteq D}{C \precsim D}.$$

The derivation of Sup is straightforward: remember that $C \precsim C$ holds (Ref), assume $C \sqsubseteq D$ and then apply (RW):

$$(\mathrm{S})\ \frac{C \precsim D}{\top \precsim \neg C \sqcup D}.$$

To see that (S) holds, assume $C \precsim D$ and note that $\models D \sqsubseteq \neg C \sqcup D$; we derive by (RW) $C \precsim \neg C \sqcup D$. Since $\models \neg C \sqsubseteq \neg C \sqcup D$, we obtain $\neg C \precsim \neg C \sqcup D$ by (Sup). Then apply (Or) to $C \precsim \neg C \sqcup D$ and $\neg C \precsim \neg C \sqcup D$, obtaining $\top \precsim \neg C \sqcup D$.

Now, we have to prove that if $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{D}^*} \sqsubseteq C$, then $\mathcal{KB} \models_{mod} \top \precsim C$.

From Lemma 12, we know that $\mathcal{T}^* \cup \mathcal{D}^*$ is in the modular consequences of $\mathcal{KB}$. Applying (S) to all DCIs $C \precsim D$ in $\mathcal{D}^*$, we have $\mathcal{KB} \models_{mod} \top \precsim \neg C \sqcup D$ from each of them. Applying (And) to all these DCIs, we have $\top \precsim \bigsqcap \overline{\mathcal{D}'}$ and, by (RW'), we obtain $\top \precsim C$.

The only-if part is an immediate consequence of Lemma 44. □

Lemma 47 (∗). *For every $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and every $C \in \mathcal{L}$, $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$ iff $\mathsf{rk}(C) = \infty$.*

Proof. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and transform it into a modularly equivalent knowledge base $\mathcal{D}'$ composed of only DCIs (see Lemma 2). Since the model $\mathcal{O}$ of the rational closure of $\mathcal{KB}$ must also be a model of $\mathcal{D}'$, we can easily derive from Lemma 9 that $\mathcal{KB} \models_{rat} C \precsim \bot$ (that is, $\mathsf{rank}_{\mathcal{KB}}(C) = \infty$) iff $\mathcal{KB} \models_{mod} C \precsim \bot$. From Lemma 45, we have that $\mathcal{KB} \models_{mod} C \precsim \bot$ iff $\mathsf{rk}(C) = \infty$, hence the result. □

Lemma 13. *For every defeasible knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and every $C \in \mathcal{L}$, $\mathsf{rank}_{\mathcal{KB}}(C) = \mathsf{rk}(C)$.*

Proof. From Lemmas 45 and 47 and Lemma 12, we can see that, given a knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ (possibly with an empty $\mathcal{T}$), we can define a modularly equivalent knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ such that all the classical information implicit in $\mathcal{D}$ is moved into $\mathcal{T}^*$. $\mathcal{KB}^*$ can be defined identifying the elements of $\mathcal{D}$ that have ∞ as ranking value, and Lemma 47 shows that w.r.t. the value ∞, $\mathsf{rank}_{\mathcal{KB}}(\cdot)$ and $\mathsf{rk}(\cdot)$ are equivalent, while Lemma 12 tells us that $\mathcal{KB}$ and $\mathcal{KB}^*$ are modularly equivalent. Once we have defined $\mathcal{KB}^*$, Lemma 46 implies that a concept $C \in \mathcal{L}$ is exceptional w.r.t. $\models_{rat}$ ($\mathcal{KB} \models_{mod} \top \precsim \neg C$) iff $\mathcal{KB} \vdash_{rat} \top \precsim \neg C$. Hence, the two ranking functions $\mathsf{rank}_{\mathcal{KB}}(\cdot)$ and $\mathsf{rk}(\cdot)$ give back exactly the same results. □

Theorem 5. *Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and let $C, D \in \mathcal{L}$. Then $\mathcal{KB} \vdash_{rat} C \precsim D$ iff $\mathcal{KB} \models_{rat} C \precsim D$.*

Proof. Since we have already proven Lemma 13, here we can use $\mathsf{rk}(\cdot)$ to indicate indifferently the equivalent ranking functions $\mathsf{rank}_{\mathcal{KB}}(\cdot)$ and $\mathsf{rk}(\cdot)$.

For the only-if part, assume $\mathcal{KB} \models_{rat} C \precsim D$. That means that either $\mathsf{rk}(C \sqcap \neg D) > \mathsf{rk}(C)$ or $\mathsf{rk}(C) = \infty$. In the first case, it means that there is some i, $0 \leq i \leq n$, such that $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg C$ does not hold and $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg(C \sqcap \neg D)$, hence $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$, i.e., $\mathcal{KB} \vdash_{rat} C \precsim D$. In the second case, we have $\mathcal{T}^* \models C \sqsubseteq \bot$, which implies $\mathcal{KB} \vdash_{rat} C \precsim D$.

For the if part, assume $\mathcal{KB} \vdash_{rat} C \precsim D$. Then either there is some i that is the lowest number such that $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg C$ does not hold (hence $\mathsf{rk}(C) = i$), or $\mathcal{T}^* \models C \sqsubseteq \bot$. In the first case, we have also that $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq D$, which implies $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqsubseteq \neg(C \sqcap \neg D)$, i.e., $\mathsf{rk}(C \sqcap \neg D) > i$. In the second case, $\mathsf{rk}(C) = \infty$, which implies $\mathcal{KB} \models_{rat} C \precsim D$. □

Corollary 4. *Checking rational entailment is* EXPTIME*-complete.*

Proof. Observe that function RationalClosure$(\cdot)$ performs at most $n + 2$ (classical) subsumption checks, where n is the number of ranks assigned to elements of $\mathcal{D}$. So the number of subsumption checks performed by function RationalClosure$(\cdot)$ is $O(|\mathcal{D}|)$. Furthermore, we need to call function ComputeRanking$(\cdot)$ to obtain the knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ and the sequence $\mathcal{E}_0, \ldots, \mathcal{E}_n$, which are needed as input to function RationalClosure$(\cdot)$. First bear in mind that function Exceptional$(\cdot)$, with $\mathcal{E}$ as input, performs at most $|\mathcal{E}|$ classical subsumption checks. From this, and an analysis of function ComputeRanking$(\cdot)$, it follows that the number of subsumption checks performed by function ComputeRanking$(\cdot)$ is $O(|\mathcal{D}|^3)$. Since we know that subsumption checking w.r.t. general TBoxes in $\mathcal{ALC}$ is EXPTIME-complete [1, Chapter 3], the result follows. □

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