Lexichographic Closure for Defeasible Description Logics

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Abstract. In the field of non-monotonic logics, the *lexicographic closure* is acknowledged as a a powerful and logically well-characterized approach; we are going to see that such a construction can be applied in the field of Description Logics, an important knowledge representation formalism, and we shall provide a simple decision decision procedure.

1 Introduction

The application of non-monotonic reasoning (see, *e.g.* [12]) to Description Logics (DLs) [1] has received a lot of attention in the last years, resulting in the development of various proposals, such as [2, 4, 6, 5, 7–11, 13–15, 21, 22]. However, between them just a few ([8–10, 14, 22]) take under consideration the preferential approach (see [19]),a well-known approach to non-monotonic reasoning. Here we take under consideration one of the main proposals in the preferential area, the *lexicographic closure* [18], developed by Lehmann for propositional languages, and never taken under consideration in the DL field. We are going to readapt such a procedure to $\mathcal{ALC}$, a significant and expressive representative of the various DLs. The procedure we are going to implement is obtained modifying the rational closure construction defined for $\mathcal{ALC}$ in [9].

We proceed as follows: we present a construction of the lexicographic closure at the propositional level that slightly generalize the construction presented by Lehmann, and still based on classical entailment tests only; then we implement such a construction in $\mathcal{ALC}$. Due to the space limits, we shall omit the proofs of the presented propositions.

2 Propositional Lexicographic Closure

Let ℓ be a finitely generated classical propositional language, defined in the usual way using $\neg, \wedge, \vee, \rightarrow$ as connectives, $C, D, \ldots$ as sentences, $\Gamma, \Delta, \ldots$ as finite sets of sentences, $\top$ and $\bot$ as abbreviations for, respectively, $A \vee \neg A$ and $A \wedge \neg A$ for some A. The symbols $\models$ and $\mid\!\sim$ will indicate, respectively, the classical consequence relation and a defeasible inference relation. An element of a consequence relation, $\Gamma \models C$ or $\Gamma \mid\!\sim C$, will be called a *sequent*, and in the case of $\mid\!\sim$ it has to be read as 'If Γ, then typically C'.

We call *conditional knowledge base* a pair $\langle \mathcal{T}, \mathcal{B} \rangle$, where $\mathcal{T}$ is a set of *strict* sequents $C \models D$, representing certain knowledge, and $\mathcal{B}$ is a set of *defeasible* sequents $C \mid\!\sim D$, representing default information.

Example 1. The typical 'penguin' example can be encoded as [3]: $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$ with $\mathcal{T} = \{P \models B\}$ and $\mathcal{B} = \{P \mid\sim \neg F, B \mid\sim F\}$. □

In what follows we are going to present a slight generalization of Lehmann's procedure [18], that is, a non-monotonic reasoning procedure that relies on a decision procedure for $\models$ only. We then suggest how to transpose such an approach into the framework of DLs.

In the present section we shall proceed in the following way: first, we define the notion of *rational consequence relation* (see *e.g.* [17]) and we present the notions of *rational* and *lexicographic closure*; then, we describe a procedure to build a *lexicographic closure* of a conditional knowledge base.

Rational Consequence Relations. The preferential approach is mainly based the identification of the structural properties that a well-behaved (both from the intuitive and the logical point of view) non-monotototonic consequence relation should satisfy. Between the possible interesting options, a particular relevance has been given to the group of properties characterizing the class of *rational consequence relations* (see [17]). A consequence relation $\mid\sim$ is *rational* iff it satisfies the following properties:

$$(\mathrm{REF})\quad C \mid\sim C \qquad \text{Reflexivity}$$

$$(\mathrm{CT})\quad \frac{C \mid\sim D \quad C \wedge D \mid\sim F}{C \mid\sim F} \qquad \text{Cut (Cumulative Trans.)}$$

$$(\mathrm{RW})\quad \frac{C \mid\sim D \quad D \models F}{C \mid\sim F} \qquad \text{Right Weakening}$$

$$(\mathrm{CM})\quad \frac{C \mid\sim D \quad C \mid\sim F}{C \wedge D \mid\sim F} \qquad \text{Cautious Monotony}$$

$$(\mathrm{OR})\quad \frac{C \mid\sim F \quad D \mid\sim F}{C \vee D \mid\sim F} \qquad \text{Left Disjunction}$$

$$(\mathrm{LLE})\quad \frac{C \mid\sim F \quad \models C \leftrightarrow D}{D \mid\sim F} \qquad \text{Left Logical Equival.}$$

$$(\mathrm{RM})\quad \frac{C \mid\sim F \quad C \not\mid\sim \neg D}{C \wedge D \mid\sim F} \qquad \text{Rational Monotony}$$

We refer the reader to [19] for an insight of the meaning of such rules. (RM) is generally considered as the strongest form of monotonicity we can use in the characterization of a reasoning system in order to formalise a well-behaved form of defeasible reasoning. The kind of reasoning we want to implement applies at the level of sequents: let $\mathcal{B} = \{C_1 \mid\sim E_1, \ldots, C_n \mid\sim E_n\}$ be a conditional knowledge base; we want the agent to be able to reason about the defeasible information at its disposal, that is, to be able to derive new sequents from his conditional base.

Semantically, rational consequence relations can be characterized by means of a particular kind of possible-worlds model, that is, ranked preferential models, but we shall not deepen the connection with such a semantical characterization here (see [17]). Except for (RM), all the above properties are *closure properties*, that is, they are preserved under intersection of the consequence relations, allowing for the definition of a notion of entailment (see [16]), called *preferential closure*; on the other hand, (RM) is not preserved under intersection, and not every rational consequence relation describes a desirable and intuitive form of reasoning. Two main constructions have been proposed to define interesting and well-behaved rational consequence relations: the rational closure in [17], and the lexicographic closure in [18].

Lehmann and Magidor's *rational closure* operation behaves in an intuitive way and it is logically strongly characterized (we refer the reader to [17, 9] for a description of

[3] Read B as 'Bird', P as 'Penguin' and F as 'Flying'.

rational closure). Notwithstanding, rational closure is considered a too weak form of reasoning, since often we cannot derive intuitive conclusions from our premises. The main problem of the rational closure is that if an individual falls under an atypical subclass (for example, *penguins* are an atypical subclass of *birds*, since they do not fly), we cannot derive anymore any of the typical properties characterizing the superclass.

Example 2. Consider the KB in example 1, and add to the set $\mathcal{B}$ the sequent $B{\mid\!\sim} W$ (read W as 'Has wings'). Even if it would be desirable to conclude that penguins have wings ($P{\mid\!\sim} W$), in the rational closure it is not possible, since penguins, being atypical birds, are not allowed to inherit *any* of the typical properties of birds.

Rational closure fails to respect the *presumption of independence* ([18], pp.63-64): even if a class does not satisfy a typical property of a superclass, we should presume that it behaves in a typical way w.r.t. the other properties, if not forced to conclude the contrary. In an attempt to overcome such a shortcoming, Lehmann has proposed in [18] a possible extension of rational closure, such to preserve the desirable logical properties of rational closure, but augmenting in an intuitive way its inferential power.

Lexicographic Closure. In this paragraph we are going to present the procedure to define the lexicographic closure of a conditional knowledge base. Our procedure just slightly generalizes the one in [18], taking under consideration also knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$ with a strict part $\mathcal{T}$.

The essence of the procedure to build the lexicographic closure of $\mathcal{K}$ consists in transforming $\langle \mathcal{T}, \mathcal{B} \rangle$ into a KB $\langle \Phi, \Delta \rangle$, where Φ and Δ are sets containing formulae instead of sequents; that is, Φ contains what we are informed to be *necessarily* true, while Δ contains the formulae we consider to be *typically*, but not necessarily true. Constructed the pair $\langle \Phi, \Delta \rangle$, we can easily define from it the lexicographic closure of $\mathcal{K}$.

So, consider $\langle \mathcal{T}, \mathcal{B} \rangle$, with $\mathcal{B} = \{C_1 {\mid\!\sim} E_1, \dots, C_n {\mid\!\sim} E_n\}$. The steps for the construction of $\langle \Phi, \Delta \rangle$ (obtained combining the construction in [18] with some results from [3]) are the following:

Step 1. We transfer the information in $\mathcal{T}$ into correspondent ${\mid\!\sim}$-sequents and add it to $\mathcal{B}$, that is, we move from a characterization $\langle \mathcal{T}, \mathcal{B} \rangle$ to $\langle \emptyset, \mathcal{B}' \rangle$, where $\mathcal{B}' = \mathcal{B} \cup \{(C \wedge \neg D) {\mid\!\sim} \bot \mid C \models D \in \mathcal{T}\}$. Intuitively, $C \models D$ is equivalent to saying that its negation is an absurdity ($(C \wedge \neg D) {\mid\!\sim} \bot$) (see [3], Section 6.5).

Step 2. We define $\Delta_{\mathcal{B}'}$ as the set of the *materializations* of the sequents in $\mathcal{B}'$, *i.e.* the material implications corresponding to such sequents: $\Delta_{\mathcal{B}'} = \{C \to D \mid C {\mid\!\sim} D \in \mathcal{B}'\}$. Also, we indicate by $\mathfrak{A}_{\mathcal{B}'}$ the set of the antecedents of the sequents in $\mathcal{B}'$: $\mathfrak{A}_{\mathcal{B}'} = \{C \mid C {\mid\!\sim} D \in \mathcal{B}'\}$.

Step 3. Now we define an *exceptionality ranking* of sequents w.r.t. $\mathcal{B}'$.

Exceptionality: Lehmann and Magidor call a formula C *exceptional* for a set of sequents $\mathcal{D}$ iff it is false in all the most typical situations satisfying $\mathcal{D}$ (see [17], Section 2.6); in particular, C is exceptional w.r.t. $\mathcal{D}$ if we can derive $\top {\mid\!\sim} \neg C$ from the preferential closure of $\mathcal{D}$. $C {\mid\!\sim} D$ is said to be exceptional for $\mathcal{D}$ iff its antecedent C is exceptional for $\mathcal{D}$. Exceptionality of sequents can be decided based on $\models$ only (see [17], Corollary 5.22), as C is exceptional for a set of sequents $\mathcal{D}$ iff $\Delta_{\mathcal{D}} \models \neg C$.

Given a set of sequents $\mathcal{D}$, indicate by $E(\mathfrak{A}_{\mathcal{D}})$ the set of the antecedents that result exceptional w.r.t. $\mathcal{D}$, that is $E(\mathfrak{A}_{\mathcal{D}}) = \{C \in \mathfrak{A}_{\mathcal{D}} \mid \Gamma_{\mathcal{D}} \models \neg C\}$, and with $E(\mathcal{D})$ the exceptional sequents in $\mathcal{D}$, *i.e.* $E(\mathcal{D}) = \{C{\mid\!\sim} D \in \mathcal{D} \mid C \in E(\mathfrak{A}_{\mathcal{D}})\}$. Obviously, for every $\mathcal{D}$, $E(\mathcal{D}) \subseteq \mathcal{D}$.

Step 3.1. We can construct iteratively a sequence $\mathcal{E}_0, \mathcal{E}_1 \ldots$ of subsets of the conditional base $\mathcal{B}'$ in the following way: $\mathcal{E}_0 = \mathcal{B}'$, $\mathcal{E}_{i+1} = E(\mathcal{E}_i)$. Since $\mathcal{B}'$ is a finite set, the construction will terminate with an empty set ($\mathcal{E}_n = \emptyset$) or a fixed point of E.

Step 3.2. Using such a sequence, we can define a ranking function r that associates to every sequent in $\mathcal{B}'$ a number, representing its level of exceptionality:

$$r(C{\mid\!\sim} D) = \begin{cases} i & \text{if } C{\mid\!\sim} D \in \mathcal{E}_i \text{ and } C{\mid\!\sim} D \notin \mathcal{E}_{i+1} \\ \infty & \text{if } C{\mid\!\sim} D \in \mathcal{E}_i \text{ for every } i\ . \end{cases}$$

Step 4. In Step 3, we defined the materialization of $\mathcal{B}'$ and the rank of every sequent in it. Now,

Step 4.1. we can determine if $\mathcal{B}'$ is inconsistent. A conditional base is inconsistent if in its preferential closure we obtain the sequent $\top{\mid\!\sim}\bot$ (from this sequent we can derive any other sequent using RW and CM). Given the result in Step 3.1, we can check the consistency of $\mathcal{B}'$ using $\Delta_{\mathcal{B}'}$: $\top{\mid\!\sim}\bot$ is in the preferential closure of $\mathcal{B}'$ iff $\Delta_{\mathcal{B}'} \models \bot$.

Step 4.2. Assuming $\mathcal{B}'$ is consistent, and given the ranking, we define the *background theory* $\widetilde{\mathcal{T}}$ of the agent as $\widetilde{\mathcal{T}} = \{\top \models \neg C \mid C{\mid\!\sim} D \in \mathcal{B}' \text{ and } r(C{\mid\!\sim} D) = \infty\}$[4] , and a correspondent set of formulae $\Phi = \{\neg C \mid \top \models \neg C \in \widetilde{\mathcal{T}}\}$ (one may verify that, modulo classical logical equivalence, $\mathcal{T} \subseteq \widetilde{\mathcal{T}}$).

Step 4.3. once we have $\widetilde{\mathcal{T}}$, we can also identify the set of sequents $\widetilde{\mathcal{B}}$, *i.e.*, the defeasible part of the information contained in $\mathcal{B}'$: $\widetilde{\mathcal{B}} = \{C{\mid\!\sim} D \in \mathcal{B}' \mid r(C{\mid\!\sim} D) < \infty\}$ (one may verify that $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$). We indicate the ranking of $\widetilde{\mathcal{B}}$ as the highest ranking value of the conditionals in $\widetilde{\mathcal{B}}$, *i.e.* $r(\widetilde{\mathcal{B}}) = \max\{r(C{\mid\!\sim} D) \mid C{\mid\!\sim} D \in \widetilde{\mathcal{B}}\}$.

Essentially, so far we have moved the non-defeasible knowledge 'hidden' in $\mathcal{B}$ to $\mathcal{T}$.

Step 5. Now we can define the lexicographic closure of $\langle\widetilde{\mathcal{T}}, \widetilde{\mathcal{B}}\rangle$. Consider the set of the *materializations* of the sequents in $\widetilde{\mathcal{B}}$, $\Delta = \{C \to D \mid C{\mid\!\sim} D \in \widetilde{\mathcal{B}}\}$, and assume that $r(\widetilde{\mathcal{B}}) = k$. Δ^k represents the subset of Δ composed by conditionals of rank k, *i.e.* $\Delta^i = \{C \to D \in \Delta \mid r(C{\mid\!\sim} D) = i\}$. We can associate to every subset $\mathcal{D}$ of Δ a string of natural numbers $\langle n_0, \ldots, n_k\rangle_{\mathcal{D}}$, where $n_0 = \mid \mathcal{D} \cap \Delta^k \mid$, and, in general, $n_i = \mid \mathcal{D} \cap \Delta^{k-i} \mid$. In such a way we obtain a string of numbers expressing how many materializations of defaults are contained in $\mathcal{D}$ for every ranking value. We can order linearly the tuples $\langle n_0, \ldots, n_k\rangle_{\mathcal{D}}$ using the classic lexicographic order: $\langle n_0, \ldots, n_k\rangle \geq \langle m_0, \ldots, m_k\rangle$ iff
(i) for every i ($0 \leq i \leq k$), $n_i \geq m_i$, or
(ii) if $n_i < m_i$, there is a j s.t. $j > i$ and $n_j > m_j$.

[4] One may easily verify the correctness of this definition referring to the following results in [3], Section 7.5.3: Definition 7.5.1, the definition of *clash* (p.178), Corollary 7.5.7, Definition 7.5.2, and Lemma 7.5.5. It suffices to show that the set of the sequents with ∞ as ranking value represents the greatest clash of $\mathcal{B}$, which can be proved quite immediately by the definition of the exceptionality ranking.

This lexicographic order is based on the intuitions that the more conditionals are satisfied, the better it is, and that more exceptional conditionals have the priority over less exceptional ones. The linear ordering $>$ between the tuples corresponds to a modular ordering $\prec$ between the subsets of Δ:

Seriousness ordering $\prec$ ([18], Definition 2). Let $\mathcal{D}$ and $\mathcal{E}$ be subsets of Δ. $\mathcal{D}$ is preferred to (*more serious than*) $\mathcal{E}$ ($\mathcal{D} \prec \mathcal{E}$) iff $\langle n_0, \dots, n_k \rangle_{\mathcal{D}} > \langle n_0, \dots, n_k \rangle_{\mathcal{E}}$.

Step 6. Given the background theory Φ and the default set Δ, we associate to the agent the pair $\langle \Phi, \Delta \rangle$ according to the steps defined so far.

Given a conditional knowledge base $\langle \mathcal{T}, \mathcal{B} \rangle$ (transformed into $\langle \Phi, \Delta \rangle$) and a set of premises Γ, we indicate by $\mathfrak{D}$ the set of the preferred subsets of Δ that are consistent with the certain information we have at our disposal, that is Φ and Γ:

$$\mathfrak{D}_\Gamma = \min_{\prec} \{ \mathcal{D} \subseteq \Delta \mid \Gamma \cup \Phi \not\models \bot \}$$

The consequence relation $\mid\!\sim^l_{\langle \mathcal{T}, \mathcal{B} \rangle}$, corresponding to the lexicographic closure of $\langle \mathcal{T}, \mathcal{B} \rangle$, will be defined as:

$$\Gamma \mid\!\sim^l_{\langle \mathcal{T}, \mathcal{B} \rangle} E \text{ iff } \Gamma \cup \Phi \cup \mathcal{D} \models E \text{ for every } \mathcal{D} \in \mathfrak{D}_\Gamma.$$

The procedure proposed by Lehmann relies heavily on a proposal by Poole, [20], but it makes also use of the cardinality and the exceptionality ranking of the sets of defaults. At the propositional level, the problem of deciding if a sequent $C \mid\!\sim D$ is in the lexicographic closure of a conditional base $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$ is exponential (see [18], p.81).

Example 3. Consider the knowledge base in the example 2: $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$ with $\mathcal{T} = \{P \models B\}$ and $\mathcal{B} = \{P \mid\!\sim \neg F, B \mid\!\sim F, B \mid\!\sim W\}$. We define **(Step 1)** the set $\mathcal{B}' = \{P \wedge \neg B \mid\!\sim \bot, P \mid\!\sim \neg F, B \mid\!\sim F, B \mid\!\sim W\}$, and the ranking values obtained from the materialization of $\mathcal{B}'$ are $r(B \mid\!\sim F) = r(B \mid\!\sim W) = 0$, $r(P \mid\!\sim \neg F) = 1$, $r(P \wedge \neg B \mid\!\sim \bot) = \infty$ **(Steps 2-3)** . Hence, we end up with a pair $\langle \Phi, \Delta \rangle$, with $\Phi = \{\neg (P \wedge \neg B)\}$, $\Delta = \Delta^0 \cup \Delta^1$, $\Delta^0 = \{B \rightarrow F, B \rightarrow W\}$, and $\Delta^1 = \{P \rightarrow \neg F\}$ **(Steps 4-5)**. We want to check if we can derive $P \mid\!\sim W$, impossible to derive from the rational closure of $\mathcal{K}$. We have to find which are the most serious subsets of Δ that are consistent with P and Φ: it turns out that there is only one, $\mathcal{D} = \{B \rightarrow W, P \rightarrow \neg F\}$, and we have $\{P\} \cup \Phi \cup \mathcal{D} \models W$, that is, $P \mid\!\sim^l_{\langle \mathcal{T}, \mathcal{B} \rangle} W$.

3 Lexicographic Closure in $\mathcal{ALC}$

Now we redefine Lehmann's procedure for the DL case. We consider a significant DL representative, namely $\mathcal{ALC}$ (see e.g. [1], Chap. 2). $\mathcal{ALC}$ corresponds to a fragment of first order logic, using monadic predicates, called *concepts*, and diadic ones, called *roles*. To ease the reader in taking account of the correspondence between the procedure presented in Section 2 and the proposal in $\mathcal{ALC}$, we are going to use the same notation for the components playing an analogous role in the two construction: capital letters $C, D, E, \dots$ now indicate *concepts*, instead of propositions, and $\models$ and $\mid\!\sim$ to indicate, respectively, the 'classical' consequence relation of $\mathcal{ALC}$ and a non-monotonic consequence relation in $\mathcal{ALC}$. We have a finite set of *concept names* $\mathcal{C}$, a finite set of *role names* $\mathcal{R}$ and the set $\mathcal{L}$ of $\mathcal{ALC}$ *-concepts* is defined inductively as follows: (i) $\mathcal{C} \subset \mathcal{L}$; (ii) $\top, \bot \in \mathcal{L}$; (iii) $C, D \in \mathcal{L} \Rightarrow C \sqcap D, C \sqcup D, \neg C \in \mathcal{L}$; and (iii) $C \in \mathcal{L}, R \in \mathcal{R} \Rightarrow \exists R.C, \forall R.C \in \mathcal{L}$. Concept $C \to D$ is used as a shortcut of $\neg C \sqcup D$. The symbols $\sqcap$ and $\sqcup$ correspond, respectively, to the conjunction $\wedge$ and the disjunction $\vee$ of classical logic. Given a set of *individuals* $\mathcal{O}$, an *assertion* is of the form $a{:}C$ ($C \in \mathcal{L}$) or of the form $(a,b){:}R$ ($R \in \mathcal{R}$), respectively indicating that the individual a is an instance of concept C, and that the individuals a and b are connected by the role R. A *general inclusion axiom* (GCI) is of the form $C \sqsubseteq D$ ($C, D \in \mathcal{L}$) and indicates that any instance of C is also an instance of D. We use $C = D$ as a shortcut of the pair of $C \sqsubseteq D$ and $D \sqsubseteq C$.

From a FOL point of view, concepts, roles, assertions and GCIs, may be seen as formulae obtained by the following transformation

$$
\begin{array}{ll@{\qquad}ll}
\tau(a{:}C) & = \tau(a,C) & & \\
\tau((a,b){:}R) & = R(a,b) & \tau(x, C \sqcap D) & = \tau(x,C) \wedge \tau(x,D) \\
\tau(C \sqsubseteq D) & = \forall x.\tau(x,C) \to \tau(x,D) & \tau(x, C \sqcup D) & = \tau(x,C) \vee \tau(x,D) \\
\tau(x,A) & = A(x) & \tau(x, \exists R.C) & = \exists y.R(x,y) \wedge \tau(y,C) \\
\tau(x,\neg C) & = \neg\tau(x,C) & \tau(x, \forall R.C) & = \forall y.R(x,y) \to \tau(y,C)\,.
\end{array}
$$

Now, a classical knowledge base is defined by a pair $\mathcal{K} = \langle \mathcal{A}, \mathcal{T} \rangle$, where $\mathcal{T}$ is a finite set of GCIs (a *TBox*) and $\mathcal{A}$ is a finite set of assertions (the *ABox*), whereas a *defeasible knowledge base* is represented by a triple $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{B} \rangle$, where additionally $\mathcal{B}$ is a finite set of *defeasible inclusion axioms* of the form $C \mathrel{\sqsubset\!\!\!\sim} D$ ('an instance of a concept C is typically an instance of a concept D'), with $C, D \in \mathcal{L}$.

Example 4. Consider Example 3. Just add a role $Prey$ in the vocabulary, where a role instantiation $(a,b){:}Prey$ is read as 'a preys on b', and add also two more concepts, I (Insect) and Fi (Fish). A defeasible KB is $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{B} \rangle$ with $\mathcal{A} = \{a{:}P, b{:}B, (a,c){:}Prey, (b,c){:}Prey\}$; $\mathcal{T} = \{P \sqsubseteq B, I \sqsubseteq \neg Fi\}$ and $\mathcal{B} = \{P \mathrel{\sqsubset\!\!\!\sim} \neg F, B \mathrel{\sqsubset\!\!\!\sim} F, B \mathrel{\sqsubset\!\!\!\sim} W, P \mathrel{\sqsubset\!\!\!\sim} \forall Prey.Fi, B \mathrel{\sqsubset\!\!\!\sim} \forall Prey.I\}$. □

The particular structure of a defeasible KB allows for the 'isolation' of the pair $\langle \mathcal{T}, \mathcal{B} \rangle$, that we could call the *conceptual system* of the agent, from the information about the individuals (formalised in $\mathcal{A}$) that will play the role of the facts known to be true. In the next section we are going to work with the information about concepts $\langle \mathcal{T}, \mathcal{B} \rangle$ first, exploiting the immediate analogy with the homonymous pair of Section 2, then we will address the case involving individuals as well.

Construction of the Lexicographic Closure. We apply to $\langle \mathcal{T}, \mathcal{B} \rangle$ a procedure analogous to the propositional one, in order to obtain from $\langle \mathcal{T}, \mathcal{B} \rangle$ a pair $\langle \Phi, \Delta \rangle$, where Φ and Δ are two sets of concepts, the former representing the background knowledge, that is, concepts necessarily applying to each individual, the latter playing the role of defaults, that is, concepts that, modulo consistency, apply to each individual. Hence, starting with $\langle \mathcal{T}, \mathcal{B} \rangle$, we apply the following steps.

Step 1. Define $\mathcal{B}' = \mathcal{B} \cup \{C \sqcap \neg D \mathrel{\sqsubset\!\!\!\sim} \bot \mid C \sqsubseteq D \in \mathcal{T}\}$. Now our agent is characterized by the pair $\langle \emptyset, \mathcal{B}' \rangle$.

Step 2. Define $\Delta_{\mathcal{B}'} = \{\top \sqsubseteq C \to D \mid C \mathrel{\sqsubset\!\!\!\sim} D \in \mathcal{B}'\}$, and define a set $\mathfrak{A}_{\mathcal{B}'}$ as the set of the antecedents of the conditionals in $\mathcal{B}'$, *i.e.* $\mathfrak{A}_{\mathcal{B}'} = \{C \mid C \mathrel{\sqsubset\!\!\!\sim} D \in \mathcal{B}'\}$.

Step 3. We determine the exceptionality ranking of the sequents in $\mathcal{B}'$ using the set of the antecedents $\mathfrak{A}_{\mathcal{B}'}$ and the materializations in $\Delta_{\mathcal{B}'}$, where a concept C is *exceptional* w.r.t. a set of sequents $\mathcal{D}$ iff $\Delta_{\mathcal{D}} \models \top \sqsubseteq \neg C$. The steps are the same of the propositional case (**Steps 3.1 – 3.2**), we just replace the expression $\Delta_{\mathcal{D}} \models \neg C$ with the expression $\Delta_{\mathcal{D}} \models \top \sqsubseteq \neg C$. In this way we define a ranking function r.

Step 4. From $\Delta_{\mathcal{B}'}$ and the ranking function r we obtain (i) that (**Step 4.1.**) we can verify if the conceptual system of the agent is consistent by checking the consistency of $\Delta_{\mathcal{B}'}$, and (ii) (**Steps 4.2.-4.3.**) we can define the real background theory and the defeasible information of the agent, respectively the sets $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{B}}$ as:

$$\widetilde{\mathcal{T}} = \{\top \sqsubseteq \neg C \mid C \mathrel{\widetilde{\sqsubseteq}} D \in \mathcal{B}' \text{ and } r(C \mathrel{\widetilde{\sqsubseteq}} D) = \infty\}$$
$$\widetilde{\mathcal{B}} = \{C \mathrel{\widetilde{\sqsubseteq}} D \mid C \mathrel{\widetilde{\sqsubseteq}} D \in \mathcal{B}' \text{ and } r(C {\mid\!\sim} D) < \infty\} \ .$$

From $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{B}}$ we define the correspondent sets of concepts $\Phi = \{\neg C \mid \top \sqsubseteq \neg C \in \widetilde{\mathcal{T}}\}$ and $\Delta = \{C \rightarrow D \mid C \mathrel{\widetilde{\sqsubseteq}} D \in \widetilde{\mathcal{B}}\}$.

Step 5. Now, obtained $\langle \Phi, \Delta \rangle$ and the ranking value of the elements of $\widetilde{\mathcal{B}}$ and, consequently, of Δ (assume $r(\widetilde{\mathcal{B}}) = k$), we can determine the seriousness ordering on the subsets of Δ. The procedure is the same as for the propositional case, that is, (i) we associate to every subset $\mathcal{D}$ of Δ a string $\langle n_0, \ldots, n_k \rangle$ with $n_i = \mid \mathcal{D} \cap \Delta^{k-i} \mid$, and we obtain a lexicographic order '$>$' between the strings. Then we define the seriousness ordering $\prec$ between the subsets of Δ as

$$\mathcal{D} \prec \mathcal{E} \text{ iff } \langle n_0, \ldots, n_k \rangle_{\mathcal{D}} > \langle n_0, \ldots, n_k \rangle_{\mathcal{E}}$$

for every pair of subsets $\mathcal{D}$ and $\mathcal{E}$ of Δ.

Hence, we obtain an analogous of the procedure defined for the propositional case by substituting the conceptual system $\langle \mathcal{T}, \mathcal{B} \rangle$ with the pair $\langle \Phi, \Delta \rangle$.

Closure Operation over Concepts. Consider the pair $\langle \Phi, \Delta \rangle$. Now we specify the notion of lexicographic closure over the concepts, that is, a relation ${\mid\!\sim}^{l}_{\langle \mathcal{T}, \mathcal{B} \rangle}$ that tells us what presumably follows from a finite set of concepts Γ. Again, we define for a set of premises Γ the set of the most serious subsets of Δ that are consistent with Γ and Φ.

$$\mathfrak{D}_{\Gamma} = \min_{\prec}\{\mathcal{D} \subseteq \Delta \mid \ \not\models \bigsqcap \Gamma \sqcap \bigsqcap \Phi \sqcap \bigsqcap \mathcal{D} \sqsubseteq \bot\}$$

Obtained $\mathfrak{D}_{\Gamma}$, the lexicographic closure is defined as follows:

$$\Gamma {\mid\!\sim}^{l}_{\langle \mathcal{T}, \mathcal{B} \rangle} E \textit{ iff } \models \bigsqcap \Gamma \sqcap \bigsqcap \Phi \sqcap \bigsqcap \mathcal{D} \sqsubseteq E \textit{ for every } \mathcal{D} \in \mathfrak{D}_{\Gamma} \ .$$

We can prove two main properties characterizing the proposition lexicographic closure and respected by ${\mid\!\sim}^{l}_{\langle \mathcal{T}, \mathcal{B} \rangle}$: (i) ${\mid\!\sim}^{l}_{\langle \mathcal{T}, \mathcal{B} \rangle}$ is a rational consequence relation, and (ii) ${\mid\!\sim}^{l}_{\langle \mathcal{T}, \mathcal{B} \rangle}$ extends the rational closure.

Proposition 1. ${\mid\!\sim}_{\langle \widetilde{\mathcal{T}}, \widetilde{\mathcal{B}} \rangle}$ *is a rational consequence relation validating* $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$.

This can be shown by noting that the analogous properties of the propositional rational consequence relation are satisfied, namely:

(REF) $C {\mid\!\sim}_{\langle\mathcal{T},\Delta\rangle} C$

(LLE) $\dfrac{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E \qquad \models C = D}{D \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E}$ (RW) $\dfrac{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} D \qquad \models D \sqsubseteq E}{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E}$

(CT) $\dfrac{C \sqcap D \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E \qquad C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} D}{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E}$ (CM) $\dfrac{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E \qquad C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} D}{C \sqcap D \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E}$

(OR) $\dfrac{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E \qquad D \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E}{C \sqcup D \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} E}$ (RM) $\dfrac{C \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} D \qquad C \not\mid\!\sim_{\langle\mathcal{T},\Delta\rangle} \neg E}{C \sqcap E \mid\!\sim_{\langle\mathcal{T},\Delta\rangle} D}$

For the rational closure in $\mathcal{ALC}$, we refer to the construction presented in [9], Section 3. We indicate by $\mid\!\sim^{r}_{\langle\mathcal{T},\mathcal{B}\rangle}$ the consequence relation defined there.

Proposition 2. *The lexicographic closure extends the rational closure,* i.e. $\mid\!\sim^{r}_{\langle\mathcal{T},\mathcal{B}\rangle} \subseteq \mid\!\sim^{l}_{\langle\mathcal{T},\mathcal{B}\rangle}$ *for every pair* $\langle\mathcal{T},\mathcal{B}\rangle$.

To prove this it is sufficient to check that, given a set of premises Γ and a pair $\langle\Phi,\Delta\rangle$, each of the sets in $\mathfrak{D}_\Gamma$ classically implies the default information that would be associated to Γ in its rational closure, as defined in [9].
Let us work out the analogue of Example 3 in the DL context.

Example 5. Consider the KB of Example 4 without the ABox. Hence, we start with $\mathcal{K} = \langle\mathcal{T},\mathcal{B}\rangle$. Then $\mathcal{K}$ is changed into $\mathcal{B}' = \{P \sqcap \neg B \mathrel{\tilde{\sqsubseteq}} \bot, I \sqcap Fi \mathrel{\tilde{\sqsubseteq}} \bot, P \mathrel{\tilde{\sqsubseteq}} \neg F, B \mathrel{\tilde{\sqsubseteq}} F, B \mathrel{\tilde{\sqsubseteq}} W, P \mathrel{\tilde{\sqsubseteq}} \forall Prey.Fi, B \mathrel{\tilde{\sqsubseteq}} \forall Prey.I\}$. The set of the materializations of $\mathcal{B}'$ is $\Delta_{\mathcal{B}'} = \{\top \sqsubseteq P \wedge \neg B \rightarrow \bot, \top \sqsubseteq I \sqcap Fi \rightarrow \bot, \top \sqsubseteq P \rightarrow \neg F, \top \sqsubseteq B \rightarrow F, \top \sqsubseteq B \rightarrow W, \top \sqsubseteq P \rightarrow \forall Prey.Fi, \top \sqsubseteq B \rightarrow \forall Prey.I\}$, with $\mathfrak{A}_{\mathcal{B}'} = \{P \wedge \neg B, I \sqcap Fi, P, B\}$. Following the procedure at **Step 3**, we obtain the ranking values of every inclusion axiom in $\mathcal{B}'$: namely, $r(B \mathrel{\tilde{\sqsubseteq}} F) = r(B \mathrel{\tilde{\sqsubseteq}} W) = r(B \mathrel{\tilde{\sqsubseteq}} \forall Prey.I) = 0$; $r(P \mathrel{\tilde{\sqsubseteq}} \neg F) = r(P \mathrel{\tilde{\sqsubseteq}} \forall Prey.Fi) = 1$ and $r(P \sqcap \neg B \mathrel{\tilde{\sqsubseteq}} \bot) = r(I \sqcap Fi \mathrel{\tilde{\sqsubseteq}} \bot) = \infty$. From such a ranking, we obtain a background theory $\Phi = \{\neg(P \wedge \neg B), \neg(I \sqcap Fi)\}$, and a default set $\Delta = \Delta^0 \cup \Delta^1$, with

$$\Delta^0 = \{B \rightarrow F, B \rightarrow W, B \rightarrow \forall Prey.I\}$$
$$\Delta^1 = \{P \rightarrow \neg F, P \rightarrow \forall Prey.Fi\}\ .$$

To check if $P \mid\!\sim^{l}_{\langle\mathcal{T},\mathcal{B}\rangle} W$, we have to find the most serious subsets of Δ that are consistent with P and the concepts in Φ (*i.e.* the most serious subsets $\mathcal{D}$ of Δ s.t. $\not\models \prod \Gamma \sqcap \prod \Phi \sqcap \prod \mathcal{D} \sqsubseteq \bot$). Turns out that there is only one, $\mathcal{D} = \{P \rightarrow \neg F, P \rightarrow \forall Prey.Fi, B \rightarrow W\}$, and $\models P \sqcap \prod \Phi \sqcap \prod \mathcal{D} \sqsubseteq W$.
It is easy to check that we obtain the analogue sequents as in the propositional case and avoid the same undesirable ones. Moreover we can derive also sequents connected to the roles, such as $B \mid\!\sim^{l}_{\langle\mathcal{T},\mathcal{B}\rangle} \forall Prey.\neg Fi$ and $P \mid\!\sim^{l}_{\langle\mathcal{T},\mathcal{B}\rangle} \forall Prey.\neg I$. □

We do not have yet a proper proof, but we conjecture that the decision procedure should be EXPTIME-complete also in $\mathcal{ALC}$.

Closure Operation over Individuals. Now we will pay attention on how to apply the lexicographic closure to the ABox. Unfortunately, the application of the lexicographic closure to the ABox results really more complicated than in the case of rational closure, as presented in the last paragraph of Section 3 in [9]. Assume that we have already transformed our conceptual system $\langle \mathcal{T}, \mathcal{B} \rangle$ into a pair $\langle \widetilde{\mathcal{T}}, \widetilde{\mathcal{B}} \rangle$, and eventually into a pair $\langle \Phi, \Delta \rangle$. In particular, dealing with the ABox, we assume to start with a knowledge base $\mathcal{K} = \langle \mathcal{A}, \widetilde{\mathcal{T}}, \Delta \rangle$. We would like to infer whether a certain individual a is presumably an instance of a concept C or not. The basic idea remains to associate to every individual in $\mathcal{A}$ every default information from Δ that is consistent with our knowledge base, respecting the seriousness ordering of the subsets of Δ. As we will see, the major problem to be addressed here is that we cannot obtain anymore a unique lexicographic extension of the KB.

Example 6. Consider $\mathcal{K} = \langle \mathcal{A}, \emptyset, \Delta \rangle$, with $\mathcal{A} = \{(a, b){:}R\}$ and $\Delta = \{A \sqcap \forall R.\neg A\}$. Informally, if we apply the default to a first, we get $b{:}\neg A$ and we cannot apply the default to b, while if we apply the default to b first, we get $b{:}A$ and we cannot apply the default to a. Hence, we may have *two* extensions. □

The possibility of multiple extensions is due to the presence of the roles, that allow the transmission of information from an individual to another; if every individual was 'isolated', without role-connections, then the addition of the default information to each individual would have been a 'local' problem, treatable without considering the concepts associated to the other individuals in the dominion, and the extension of the knowledge base would have been unique. On the other hand, while considering a specific individual, the presence of the roles forces to consider also the information associated to other individuals in order to maintain the consistency of the knowledge base, and, as show in example 6, the addition of default information to one individual could prevent the association of default information to another.

We assume that $\langle \mathcal{A}, \mathcal{T} \rangle$ is consistent, *i.e.* $\langle \mathcal{A}, \mathcal{T} \rangle \not\models a{:}\bot$, for any a. For the sake of this paper, we will assume that $\mathcal{T}$ is *unfoldable*, that is defined as follows: (i) $\mathcal{T}$ contains axioms of the form $A \sqsubseteq C$ or $A = C$, where A is a concept name and C a concept; (ii) for any concept name A, there is at most one axiom having A on the left-hand side; (iii) $\mathcal{T}$ is *acyclic*, *i.e.* there is no concept name A that depends on A [5]. Besides having a high practical interest, unfoldable TBoxes have the characteristics that they can be removed in the following way: given $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \Delta \rangle$, (i) replace any inclusion axiom $A \sqsubseteq C \in \mathcal{T}$ with $A = C \sqcap A'$, where A' is a new concept name; (ii) in $\mathcal{A}$ and Δ, replace recursively any occurrence of concept names with their definition in $\mathcal{T}$; and (iii) remove $\mathcal{T}$ from $\mathcal{K}$. Hence, we may assume that $\mathcal{K}$ is of the form $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$. We may also assume that any concept in $\mathcal{A}$ is in *Negation Normal Form*, that is, a negation may occur in front of a concept name only (this is achieved in the usual way by removing double negations and pushing negation inwards [6]). Without loss of generality, we will further assume that $\mathcal{A}$ is closed under the following 'completion' rules: (i) if $a{:}C \sqcap D \in \mathcal{A}$ then both $a{:}C$ and $a{:}D$ are in $\mathcal{A}$; (ii) if $a{:}\exists R.C \in \mathcal{A}$ then there are $(a, b){:}R$ and $b{:}C$ in $\mathcal{A}$; and (iii) if $a{:}\forall R.C$ and $(a, b){:}R$ are in $\mathcal{A}$ then so is $b{:}C$. In this way, $\mathcal{A}$ contains all the information that is shared among all models of $\mathcal{A}$. Now, with $\mathcal{O}_{\mathcal{A}}$ we indicate the

[5] *A depends directly on B* iff there is an axiom in $\mathcal{T}$ having A in the left-hand side and B in the right-hand side. The relation *depends on* is defined as the transitive closure of the relation *depends directly on*.

[6] Note that $\neg\forall R.C$ is the same as $\exists R.\neg C$.

individuals occurring in $\mathcal{A}$. Given $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$, we say that a knowledge base $\widetilde{\mathcal{K}} = \langle \mathcal{A}_\Delta \rangle$ is a *default extension* of $\mathcal{K}$ iff

- $\widetilde{\mathcal{K}}$ is classically consistent and $\mathcal{A} \subseteq \mathcal{A}_\Delta$.
- For any $a \in \mathcal{O}_\mathcal{A}$, $a{:}C \in \mathcal{A}_\Delta \setminus \mathcal{A}$ iff $C = \bigsqcap \mathcal{D}$ for some $\mathcal{D} \subset \Delta$ s.t. $\mathcal{A}_\Delta \cup \{a{:}\mathcal{D}'\} \models \bot$ for every $\mathcal{D}' \subseteq \Delta$, with $\mathcal{D}' \prec \mathcal{D}$.
- There is no $\mathcal{K}' \supset \widetilde{\mathcal{K}}$ satisfying these conditions.

Essentially, we assign to each individual $a \in \mathcal{O}_\mathcal{A}$ one of the most serious default set that are consistent with the ABox.

Example 7. Referring to Example 6, consider $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$, with $\mathcal{A} = \{(a, b) : R\}$ and $\Delta = \{A \sqcap \forall R.\neg A, \top\}$. Then we have two default-assumption extensions, namely $\widetilde{\mathcal{K}}_1 = \mathcal{A} \cup \{a{:}A, a{:}\forall R.\neg A, b{:}\top\}$ and $\widetilde{\mathcal{K}}_2 = \mathcal{A} \cup \{b{:}A, b{:}\forall R.\neg A, a{:}\top\}$. □

A procedure to obtain a set A_s of default extensions is as follows:

(i) fix a linear order $s = \langle a_1, \ldots, a_m \rangle$ of the individuals in $\mathcal{O}_\mathcal{A}$, and let $\mathsf{A}_s^0 = \{\mathcal{A}\}$.

Now, for every a_i, $1 \leq i \leq m$, do:
(ii) for every element $\mathcal{X}$ of A_s^{i-1}, find the set all the $\prec$-minimal default sets $\{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$, s.t. $\mathcal{D}_j \subseteq \Delta$ and $\mathcal{X} \cup \{a_1{:}\bigsqcap \mathcal{D}_j\}$ is consistent $(1 \leq j \leq n)$;

(iii) Define a new set A_s^i containing all the sets $\mathcal{X} \cup \{a_1{:}\bigsqcap \mathcal{D}_j\}$ obtained at the point (ii).

(iv) Move to the next individual a_{i+1}.

(v) Once the points (ii)-(iv) have been applied to all the individuals in the sequence $s = \langle a_1, \ldots, a_m \rangle$, set $\mathsf{A}_s = \mathsf{A}_s^m$, where A_s^m is the final set obtained at the end of the procedure.

It can be shown that

Proposition 3. *An Abox $\mathcal{A}'$ is a default extension of $\mathcal{K} = \{\mathcal{A}, \Delta\}$ iff it is in the set A_s obtained by some linear ordering s on $\mathcal{O}_\mathcal{A}$ and the above procedure.*

For instance, related to Example 7, $\widetilde{\mathcal{K}}_1$ is obtained from the order $\langle a, b \rangle$, while $\widetilde{\mathcal{K}}_2$ is obtained from the order $\langle b, a \rangle$.

Example 8. Refer to Example 5, and let $\mathcal{K} = \{\mathcal{A}, \mathcal{T}, \Delta\}$, where $\mathcal{A} = \{a{:}P$, $b{:}B$, $(a, c){:}Prey$, $(b, c){:}Prey\}$, $\mathcal{T} = \{P = B \sqcap B', I = \neg Fi \sqcap I'\}$, $\Delta = \{B \to F, B \to W, B \to \forall Prey.I, P \to \neg F, P \to \forall Prey.Fi\}$' After expanding the TBox and 'applying' the completion rules to $\mathcal{A}$, we get $\mathcal{K} = \{\mathcal{A}, \Delta\}$, where $\mathcal{A} = \{a{:}B \sqcap B', a{:}B, a{:}B', b{:}B,$ $(a, c){:}Prey, (b, d){:}Prey\}$, $\Delta = \{B \to F, B \to W, B \to \forall Prey.(\neg Fi \sqcap I'), (B \sqcap B') \to \neg F, (B \sqcap B') \to \forall Prey.Fi)$. If we consider an order where a is considered before b then we associate $\mathcal{D} = \{B \to W, (B \sqcap B') \to \neg F, (B \sqcap B') \to \forall Prey.Fi\}$ to a, and consequently c is presumed to be a fish and we are prevented in the association of $B \to \forall Prey.(\neg Fi \sqcap I')$ to b. If we consider b before a, c is not a fish and we cannot apply $B \sqcap B') \to \forall Prey.Fi$ to a. □

If we fix a priori a linear order s on the individuals, we may define a consequence relation depending on the default extensions generated from it, *i.e.* the sets of defaults in A_s: we say that $a{:}C$ is a *defeasible consequence* of $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$ w.r.t. s, written $\mathcal{K} \Vdash^l_s a{:}C$, iff $\mathcal{A}' \models a{:}C$ for every $\mathcal{A} \in \mathsf{A}_s$.

For instance, related to Example 7 and order $s_1 = \langle a, b \rangle$, we may infer that $\mathcal{K} \Vdash^l_{s_1} a{:}A$, while with order $s_2 = \langle b, a \rangle$, we may infer that $\mathcal{K} \Vdash^l_{s_2} b{:}A$.

The interesting point of such a consequence relation is that it satisfies the properties of a *rational* consequence relation in the following way.

REF_{DL} $\quad \langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C$ for every $a{:}C \in \mathcal{A}$

$$LLE_{DL} \quad \frac{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C \quad \models D = E}{\langle \mathcal{A} \cup \{b{:}E\}, \Delta \rangle \Vdash_s a{:}C}$$

$$RW_{DL} \quad \frac{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C \quad \models C \sqsubseteq D}{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}D}$$

$$CT_{DL} \quad \frac{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A}, \Delta \rangle \Vdash_s b{:}D}{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C}$$

$$CM_{DL} \quad \frac{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A}, \Delta \rangle \Vdash_s b{:}D}{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C}$$

$$OR_{DL} \quad \frac{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A} \cup \{b{:}E\}, \Delta \rangle \Vdash_s a{:}C}{\langle \mathcal{A} \cup \{b{:}D \sqcup E\}, \Delta \rangle \Vdash_s a{:}C}$$

$$RM_{DL} \quad \frac{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A}, \Delta \rangle \nVdash_s b{:}\neg D}{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C}$$

We can show that

Proposition 4. *Given $\mathcal{K}$ and a linear order s of the individuals in $\mathcal{K}$, the consequence relation $\Vdash^l_s$ satisfies the properties* $\mathrm{REF_{DL}} - \mathrm{RM_{DL}}$.

4 Conclusions

In this paper we have proposed an extension of a main non-monotonic construction, the lexicographic closure (see [18]), for the DL $\mathcal{ALC}$. This work carries forward the approach presented in [9], where the adaptation of the rational closure in $\mathcal{ALC}$ is presented. Here we have first presented the procedure at the propositional level, and then we have adapted it for $\mathcal{ALC}$, first considering only the conceptual level, the information contained in the TBox, and then considering also the particular information about the individuals, the ABox, assuming we are working with unfoldable KB.
It is straightforward to see that, while the procedure defined for the TBox is simple and well-behaved, the procedure for the ABox is really more complex than the one for the rational closure presented in [9].
Besides checking the exact costs of these procedures from the computational point of view and checking for which other DL formalisms we can apply them, we conjecture that a semantical characterization of the above procedures can be obtained using the kind of semantical constructions presented in [8].

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