Towards Rational Closure for Fuzzy Logic: The Case of Propositional Gödel Logic

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Abstract. In the field of non-monotonic logics, the notion of *rational closure* is acknowledged as a landmark and we are going to see whether such a construction can be adopted in the context of mathematical fuzzy logic, a so far (apparently) unexplored journey. As a first step, we will characterise rational closure in the context of Propositional Gödel Logic.

1 Introduction and Motivation

A lot of attention has been dedicated to *non-monotonic* (or *defeasible*) reasoning (see, *e.g.* [11]) to accommodate reasoning patters with exceptions such as "typically, a bird flies, but a penguin is a bird that does not fly". Among of the many proposals, the notion of *rational closure* [18] is acknowledged as a landmark for non-monotonic reasoning due to its firm logical properties.

On the other hand, the main formalism developed for dealing with vague notions is represented by the class of *multi-valued* or *mathematical fuzzy* logics [15, 16], allowing to reason with statements involving vague concepts such as "a very young bird does not fly". These logics allow to associate to a statement a truth value that is chosen not only between false and true (*i.e.*, $\{0, 1\}$), but usually from the real interval $[0, 1]$ and, thus, allowing to specify statements of *graded* truth [3].

Here we propose a first attempt towards the definition of a logical system that combines such two forms of reasoning, namely reasoning about vagueness and defeasible reasoning via rational closure, allowing to cope with reasoning patterns such as

> "Typically, a ripe fruit is sweet, but a ripe bitter melon is a ripe fruit that is not sweet." [4]

More specifically, in what follows, we will propose a formalism for reasoning about non-monotonic conditionals involving fuzzy statements as antecedents and consequents, *i.e.* conditionals $C \rightsquigarrow D$ that are read as

$(*)$ "Typically, if C is true to a positive degree, then D is true to a positive degree too."

[3] The previous statement may be graded as a bird may be very young to some degree depending on the birds age.

[4] As the bitter melon ripens, the flesh (rind) becomes tougher, more bitter, and too distasteful to eat.

Note that such interpretation is different from other ones appeared in the literature, notably *e.g.* [2–4, 6, 7, 9, 10, 20, 21].

While one usually distinguishes three different fuzzy logics, namely Gödel, Product and Łukasiewicz logics [16] to interpret graded statements [5], we start the journey of our investigation with Propositional Gödel Logic, leaving the other two and extensions to (notable fragments of) First-Order Logic for future work.

Related Work. While there have been a non negligible amount of work related to the notion of rational closure in the classical logic setting, very little is know about it in the context of mathematical fuzzy logic. Somewhat related are [2–4, 6, 7, 9, 10], which rely on a possibilistic logic setting. Specifically, [2] shows that the notion of classical rational closure can be related to possibility distributions: roughly a conditional $C \rightsquigarrow D$ is interpreted as $\Pi(C \wedge D) > \Pi(C \wedge \neg D)$, *i.e.* the possibility of classical formula $C \wedge D$ is greater than the possibility of $C \wedge \neg D$. The idea has then be used later on in [4] and related works such as [3, 6, 10], however, addressing only marginally the fuzzy case as well, by proposing various interpretation of the fuzzy conditional $C \rightsquigarrow D$, *e.g.* along the paradigm "the more C the more it is certain that C implies D". This is a different interpretation as the one proposed here and, indeed, seems not to apply to the typical ripe fruits are sweet case. To the best of our knowledge, there has been no attempt so far to combine rational closure in the context of a pure mathematical fuzzy logic setting, which, however, does not mean that an approach based on possibilistic logic may not be viable in the future as well.

In the following, we proceed as follows. After introducing some preliminary notions in the next section, section 3 characterises preferential entailment, section 4 characterises rational monotony, and eventually section 5 concludes and addresses future work. The Appendix reports some proofs.

2 Preliminaries

Syntax. We start with a standard propositional language, defined from a finite set P of atomic propositions and connectives $\{\neg, \wedge, \vee, \supset, \equiv\}$. Let $\mathcal{L}$ be the set of the propositional formulae, that we indicate with $C, D, \ldots$. From $\mathcal{L}$ and the operator $\rightsquigarrow$ we define the conditionals $\mathcal{C} = \{C \rightsquigarrow D \mid C, D \in \mathcal{L}\}$, where $C \rightsquigarrow D$ is interpreted as $(*)$.

A knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ consists of a finite set $\mathcal{T}$ of propositions, indicating what the agent considers as fully true, and a finite set of conditionals $\mathcal{D}$, describing defeasible information about what typically holds.

Example 1. The example about ripe fruits are sweet, but a ripe bitter melon isn't, can be encoded as follows:

$$\mathcal{T} = \{rbm \supset (rf \wedge \neg s), rbm \supset bm\}$$
$$\mathcal{D} = \{rf \rightsquigarrow s\} ,$$

where rbm, bm, rf and s encode ripe bitter melon, bitter melon, ripe fruits and sweet, respectively. □

[5] The main reason is that any other t-norm, the function used to interpret conjunction, can be obtained as a combination of these three.

Semantics. At the base of our (preferential) semantics there are the valuations for propositional Gödel logic. A valuation u is a function that maps each atomic proposition in P into $[0, 1]$, and u is then extended inductively as follows:

$$\begin{array}{ll} u(C \wedge D) & = u(C) \otimes u(D) \\ u(C \vee D) & = u(C) \oplus u(D) \\ u(C \supset D) & = u(C) \Rightarrow u(D) \\ u(\neg C) & = \ominus u(C)\ . \end{array}$$

$C \equiv D$ is, as usual, an abbreviation for $(C \supset D) \wedge (D \supset C)$. In Gödel logic, the semantic operators are defined as:

$$\begin{array}{ll} m \otimes n & = \min(m, n) \\ m \oplus n & = \max(m, n) \\ m \Rightarrow n & = \begin{cases} 1 \text{ if } m \leq n \\ n \text{ otherwise} \end{cases} \\ \ominus m & = \begin{cases} 1 \text{ if } m = 0 \\ 0 \text{ otherwise} \end{cases} \end{array}$$

with $m, n \in [0, 1]$.

Let $\mathcal{I} = \{u, v, \ldots\}$ be the set of all the valuations for language $\mathcal{L}$. We shall indicate with $\models$ the entailment relation defined on such interpretations, where, given a finite set of propositions Γ, $\Gamma \models D$ iff for every valuation $u \in \mathcal{I}$ that verifies the premises (*i.e.*, s.t. for every proposition $C \in \Gamma$, $u(C) = 1$), it holds that $u(D) = 1$. Note that $\Gamma \models D$ can be decided *e.g.* via the Hilbert style calculi described in [16], or with more practical methods such as [1, 14]. Anyway, deciding entailment is a coNP-complete problem [16].

Properties of the Conditionals. The preferential approach to crisp non-monotonic reasoning is characterised by the satisfaction of some desirable properties.

Here we consider the relevant properties w.r.t. the material implication, instead that w.r.t. the consequence relation as usually presented in the crisp propositional case. The properties we take under consideration are *Reflexivity, Left Logical Equivalence, Right Weakening, Cumulative Transitivity (Cut), Monotony,* and *Disjunction in the Premises*, which are illustrated below.

(REF) $C \supset C$

(LLE) $\dfrac{C \supset E \quad C \equiv D}{D \supset E}$ (RW) $\dfrac{C \supset D \quad D \supset E}{C \supset E}$

(CT) $\dfrac{C \wedge D \supset E \quad C \supset D}{C \supset E}$ (MON) $\dfrac{C \supset E}{C \wedge D \supset E}$

(OR) $\dfrac{C \supset E \quad D \supset E}{C \vee D \supset E}$

It is rather straightforward to proof that

Proposition 1. *Propositional Gödel logic satisfies the properties (REF), (LLE), (RW), (CT), (MON), and (OR).*

The properties (REF), (LLE), (RW), (CT), and (OR) are interesting because they represent a set of reasonable and desirable properties for a logic-based reasoning system. On the other hand, (MON) is the property that we want to drop, still keeping a constrained form of monotony that is appropriate for reasoning about typicality. In particular, the first form of constrained monotony that we take under consideration is *Cautious Monotony* (CM). Specifically, the set of properties involving defeasible conditionals we are interested in is the following:

$$
\begin{array}{ll}
\text{(REF) } C \rightsquigarrow C & \\[1em]
\text{(LLE) } \dfrac{C \rightsquigarrow E \;\; C \equiv D}{D \rightsquigarrow E} & \text{(RW) } \dfrac{C \rightsquigarrow D \;\; D \supset E}{C \rightsquigarrow E} \\[1em]
\text{(CT) } \dfrac{C \wedge D \rightsquigarrow E \;\; C \rightsquigarrow D}{C \rightsquigarrow E} & \text{(CM) } \dfrac{C \rightsquigarrow E \;\; C \rightsquigarrow D}{C \wedge D \rightsquigarrow E} \\[1em]
\text{(OR) } \dfrac{C \rightsquigarrow E \;\; D \rightsquigarrow E}{C \vee D \rightsquigarrow E} &
\end{array}
\tag{1}
$$

The meaning of (CM) is the following: if in every *typical* situation in which C has a positive degree of truth also D has a positive degree of truth, then a typical situation for $C \wedge D$ will be a typical situation also for C, and whatever typically follows from C (*e.g.* E) follows also from $C \wedge D$. In classical logic the set of properties above identifies the class of the *preferential conditionals* [17].

Example 2 (Example 1 cont.). Consider Example 1. Let us add the defeasible information "typically, a ripe fruit tastes good" represented via the conditional

$$rf \rightsquigarrow tg \,,$$

where tg stands for "tastes good". Then, by using (CM) we may infer that

$$rf \wedge tg \rightsquigarrow s$$

i.e., "typically, a ripe and good tasting fruit is sweet". □

3 Characterising preferential entailment

Next, we want to define a very basic non-monotonic connection between the antecedent C and the consequent D according to the interpretation of conditionals given in $(*)$, that is the conditional $C \rightsquigarrow D$ indicates that in the most typical situations in which C has a positive degree of truth, also D has a positive degree of truth. However, note that Gödel implication is interpreted w.r.t. a specific connection between the truth values of the antecedent and the consequent, that is, the conditional is true if the truth value of the antecedent is at most as high as the truth value of the consequent.

As a consequence, we won't interpret the conditional $C \rightsquigarrow D$ as the truth of $C \supset D$ in the most typical situations in which C has a positive degree of truth, but we shall refer instead to the truth value of $C \supset \neg\neg D$ in the typical situations. As is easy to see from the definition of Gödel negation above, $\neg\neg D$ is true iff D has a positive truth value. Hence, the implication $C \supset \neg\neg D$ is true either if C is totally false, or if D has a positive degree of truth, and that is the kind of connection that we want to model with our conditional.

Note that in propositional Gödel logic the two implications $C \supset \neg\neg D$ and $C \supset \neg\neg\neg\neg D$ are logically equivalent; hence, in order to introduce such an interpretation of our conditional we have to introduce also a new rule (DN) (*Double Negation*), directly connected to the just mentioned logical equivalence.

$$\text{(DN)}\ \frac{C \rightsquigarrow \neg\neg D}{C \rightsquigarrow D}$$

The first system we are going to take under consideration corresponds to the class of conditionals that is characterised by the preferential properties, specified in the previous section, plus (DN). We shall read such properties as derivation rules, defining a closure operation on the knowledge bases.

More specifically, given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, we shall indicate with $\models_{\mathcal{T}}$ the consequence relation obtained from the Gödel consequence relation $\models$ adding the propositions in $\mathcal{T}$ (what the agent considers as strictly true) as extra axioms. Then we shall use the conditionals in $\mathcal{D}$, the consequence relation $\models_{\mathcal{T}}$ and all the rules in Eq. (1) and rule (DN) to define a closure operation P over the knowledge base. The *closure* $P(\mathcal{K})$ will be the set of defeasible conditionals that is derivable from $\mathcal{D}$ using these rules as derivation rules and $\models_{\mathcal{T}}$ as the underlying consequence relation. For instance, if $C \rightsquigarrow D$ is in $P(\mathcal{K})$ and $\models_{\mathcal{T}} D \supset E$, then $C \rightsquigarrow E \in P(\mathcal{K})$ by (RW).

Example 3 (Example 2 cont.). Consider Example 2. Then it can be verified that all conditionals in $\mathcal{D}$ belong to $P(\mathcal{K})$ as well as:

$$\begin{array}{l} rf \wedge tg \rightsquigarrow s \\ rf \wedge s \rightsquigarrow tg\ . \end{array}$$

□

We next are going to completely characterise such an inference relation from the semantics point of view with a class of preferential interpretations. The elements of the interpretations we are going to define will be the *believe states* $\mathcal{A}, \mathcal{B}, \ldots$, that are sets of valuations characterising a possible state of affairs that the agent can consider as true. Hence, the set of all the possible believe states will be the power-set $\mathscr{P}(\mathcal{I})$ of all the classical Gödel valuations.

Definition 1 (Belief-state interpretation). *A* belief-state interpretation *(bs-interpretation, for short) is a pair $M = \langle \mathcal{S}, \prec \rangle$, with $\mathcal{S} \subseteq \mathscr{P}(\mathcal{I})$ and $\prec$ a preferential relation between the states; $\prec$ is asymmetric and transitive and satisfies the property of smoothness (defined below).*

The meaning of $\mathcal{A} \prec \mathcal{B}$ is that the belief state $\mathcal{A}$ describes a situation that is more typical than the belief state $\mathcal{B}$.

In the following, we shall indicate with $\hat{C}$ the *extension* of C in M, that is, the set of belief states in M s.t. each valuation in the belief state associates to C a positive degree of truth, *i.e.*

$$\hat{C} = \{\mathcal{A} \in \mathcal{S} \mid u(C) > 0 \text{ for all } u \in \mathcal{A}\} \ .$$

Next, we define the set of the typical belief states of C, denoted $\overline{C}$, as the set of the preferred states in the extension of C, that is

$$\overline{C} = \min_{\prec}(\hat{C}) = \{\mathcal{A} \in \hat{C} \mid \nexists \mathcal{B} \in \hat{C} \text{ such that } \mathcal{B} \prec \mathcal{A}\} \ .$$

Now, we will use $\overline{C}$ to define the smoothness condition.

Definition 2 (Smoothness condition). *Given a preferential model $M = \langle \mathcal{S}, \prec \rangle$, the preferential relation $\prec$ satisfies the smoothness condition iff for every $C \in \mathcal{L}$, if $\hat{C} \neq \emptyset$ then $\overline{C} \neq \emptyset$.*

We are going now to use the bs-interpretations to reason about conditionals, that is, we will define a consequence relation that, given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, gives back new non-monotonic conditionals considered as valid.

Specifically, the notion that a bs-interpretation $M = \langle \mathcal{S}, \prec \rangle$ *verifies* a proposition C, denoted $M \mid\!\approx C$, is defined as follows:

$$M \mid\!\approx C \text{ iff for every } \mathcal{A} \in \mathcal{S}, \text{ for every } u \in \mathcal{A}, u(C) = 1 \ .$$

The notion that $M = \langle \mathcal{S}, \prec \rangle$ *verifies* a conditional $C \rightsquigarrow D$, denoted $M \mid\!\approx C \rightsquigarrow D$, is defined as:

$$M \mid\!\approx C \rightsquigarrow D \text{ iff for every } \mathcal{A} \in \overline{C}, \text{ for every } u \in \mathcal{A}, u \models C \supset \neg\neg D \ .$$

Hence $C \rightsquigarrow D$ is interpreted as the most typical belief states in which if C has a positive degree of truth, then D has a positive degree of truth too.

We now move on to the definition of entailment for the conditionals. Given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, we take under consideration all the bs-interpretations that verify both the propositions in $\mathcal{T}$ and the conditionals in $\mathcal{D}$. So, we say that a bs-interpretation M is a *bs-model* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ iff $M \mid\!\approx E$ for every $E \in \mathcal{T}$ and $M \mid\!\approx E \rightsquigarrow F$ for every $E \rightsquigarrow F \in \mathcal{D}$.

Definition 3 (Entailment relation $\mid\!\approx$). *A proposition C is* entailed *by $\mathcal{K}$, denoted $\mathcal{K} \mid\!\approx C$, iff for every bs-model M of $\mathcal{K}$, $M \mid\!\approx C$ holds. A conditional $C \rightsquigarrow D$ is* entailed *by $\mathcal{K}$, denoted $\mathcal{K} \mid\!\approx C \rightsquigarrow D$, iff for every bs-model M, $M \mid\!\approx C \rightsquigarrow D$ holds.*

Remark 1. It is immediate to verify that for a proposition $C, \mathcal{K} \mid\!\approx C$ iff $\mathcal{T} \models C$, *i.e.*, $\models_{\mathcal{T}} C$. Hence, a proposition C is a preferential consequence of $\mathcal{K}$ iff it is fuzzy consequence of $\mathcal{T}$.

Now we want to prove that the entailment relation $\mid\!\approx$ characterises the closure operator P, *i.e.* given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, let the closure $P(\mathcal{K})$ be the set of defeasible conditionals that are derivable from $\mathcal{D}$ using all the rules in Eq. (1) and rule (DN), then

$$P(\mathcal{K}) = \{C \rightsquigarrow D \mid \mathcal{K} \mid\!\approx C \rightsquigarrow D\} \ .$$

To do so, we next illustrate several properties that follow from the properties of the closure operation P that will be useful in our proofs. The proofs can be found in the appendix.

Lemma 1. *The conditional* $\rightsquigarrow$ *satisfies* supraclassicality *(SUPRA):*

$$\text{(SUPRA)}\frac{C \supset D}{C \rightsquigarrow D}$$

Supraclassicality describes an important property of non-monotonic reasoning, that is, whatever is derivable from $\mathcal{T}$ using propositional Gödel logic is also a defeasible consequence.

Lemma 2. *If a conditional* $\rightsquigarrow$ *satisfies the properties defining the closure operation* P*, then it satisfies also the following properties:*

$$\text{(EQUIV)}\ \frac{C \rightsquigarrow D \quad D \rightsquigarrow C \quad C \rightsquigarrow E}{D \rightsquigarrow E} \qquad \text{(AND)}\ \frac{C \rightsquigarrow D \quad C \rightsquigarrow E}{C \rightsquigarrow D \wedge E}$$

$$\text{(MPC)}\ \frac{C \rightsquigarrow D \supset E \quad C \rightsquigarrow D}{C \rightsquigarrow E} \qquad \text{(1)}\ \frac{C \vee D \rightsquigarrow C \quad C \rightsquigarrow E}{C \vee D \rightsquigarrow E}$$

$$\text{(2)}\ \frac{C \rightsquigarrow E \quad D \rightsquigarrow F}{C \vee D \rightsquigarrow E \vee F} \qquad \text{(3)}\ \frac{C \rightsquigarrow D}{C \rightsquigarrow \neg\neg D}$$

$$\text{(4)}\ \frac{C \vee D \rightsquigarrow C \quad D \vee E \rightsquigarrow D}{C \vee E \rightsquigarrow C}$$

Next, soundness is established.

Proposition 2 (Soundness). *Given a knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$*, if a conditional* $C \rightsquigarrow D$ *is in* $P(\mathcal{K})$ *then* $\mathcal{K} \mid\!\approx C \rightsquigarrow D$.

Now we proceed to the proof of the completeness. The proof uses the same general strategy of the proof in [17] [6], based on the notion of *normal valuations* (in [17] called *normal worlds*), but, since the semantic structure is different, the proof is different too.

So, first, let's define the notion of *normal valuation* for a proposition C, that is, a valuation that makes true all the conditionals in $P(\mathcal{K})$ that have C as antecedent.

Definition 4 (Normal valuation). *A valuation* u *is* normal *for a proposition* C *w.r.t a knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ *iff* $u(C) > 0$*, for every proposition* $E \in \mathcal{T}$ $u(E) = 1$*, and for every proposition* D *s.t.* $C \rightsquigarrow D \in P(\mathcal{K})$*,* $u(C \supset \neg\neg D) = 1$ *(*i.e., $u(\neg\neg D) = 1$*).*

Now, we need a main lemma, that states that taking under consideration all the normal valuation for a proposition C we are able to characterise the closure P w.r.t. C.

Lemma 3. *For every proposition* D*,* $C \rightsquigarrow D \in P(\mathcal{K})$ *iff for every valuation* u *that is normal for* C *w.r.t.* $\mathcal{K}$*,* $u(D) > 0$ *holds.*

[6] Since the conditionals in [17] are metalinguistic sequents of a non-monotonic consequence relation, there the authors present a representation result. Here, since we consider the non-monotonic conditional as a conditional of the language, we present a completeness result.

The Lemma above is the main result to prove our completeness. Furthermore, in the following if $C \rightsquigarrow D$ and $D \rightsquigarrow C$ are both in $P(\mathcal{K})$, then we denote this as $C \sim D \in P(\mathcal{K})$. The following can be shown:

Lemma 4. *$C \sim D \in P(\mathcal{K})$ iff for every proposition E, $C \rightsquigarrow E \in P(\mathcal{K})$ iff $D \rightsquigarrow E \in P(\mathcal{K})$.*

From this it follows immediately that if $C \sim D \in P(\mathcal{K})$, a valuation u is normal for C iff it is normal for D.

Given $\mathcal{K}$, we indicate with $C^{\sim}$ the set of all the propositions that are preferentially equivalent to C w.r.t. $\mathcal{K}$, namely

$$C^{\sim} = \{D \mid C \sim D\} \ .$$

Moreover, we indicate with $[C^{\sim}]$ the belief state containing exactly all the valuations that are normal for the propositions in $C^{\sim}$. Now we define an ordering of the propositional formulas w.r.t. the conditionals in the preferential closure $P(\mathcal{K})$.

Definition 5. *C is* not less ordinary *than D, denoted $C \leq D$, iff $C \vee D \rightsquigarrow C \in P(\mathcal{K})$. Furthermore, we define $C < D$ iff $C \leq D$ and $D \not\leq C$.*

The following lemma can be shown.

Lemma 5. *If $\rightsquigarrow$ is a preferential conditional, then $<$ is asymmetric and transitive.*

It is easy to see from the definitions of $C^{\sim}$ and $<$ that if C and D are preferentially equivalent, then they have the same relative position in the ordering $<$, that is:

Lemma 6. *If C and D are both in $C^{\sim}$, then for every proposition E, $C < E$ iff $D < E$ and $E < C$ iff $E < D$.*

Now we have all the ingredients to define a preferential model $M^{\mathcal{K}} = \{\mathcal{S}^{\mathcal{K}}, \prec^{\mathcal{K}}\}$ satisfying exactly the conditionals in $P(\mathcal{K})$. Specifically, let $\mathcal{S}^{\mathcal{K}}$ be the set of the belief states that correspond to all the valuations that are normal for some formula w.r.t. $\mathcal{K}$, that is

$$\mathcal{S}^{\mathcal{K}} = \{[C^{\sim}] \mid C \in L\} \ .$$

Note that $[C^{\sim}] = \emptyset$ iff $\models_{\mathcal{T}} \neg C$, while all the valuations that are present in the nonempty belief sets must verify $\mathcal{T}$, by Definition 4. Let $\prec^{\mathcal{K}}$ to be defined on $\leq$ in the following way:

$$[C^{\sim}] \prec^{\mathcal{K}} [D^{\sim}] \text{ iff } C < D \ .$$

Some properties of the interpretation $M^{\mathcal{K}}$ are easily shown:

Lemma 7. *Given $M^{\mathcal{K}}$, for every proposition C, $\overline{C} = \{[C^{\sim}]\}$.*

Lemma 8. *$M^{\mathcal{K}}$ is a preferential interpretation.*

From these lemmas it is immediate to see that $M^{\mathcal{K}}$ is a belief-state model that verifies $\mathcal{K}$, and it is exactly the model we need to prove completeness.

Lemma 9. *Given a knowledge base $\mathcal{K}$, for every conditional $C \rightsquigarrow D$, $M^{\mathcal{K}} \models\!\!\approx C \rightsquigarrow D$ implies $C \rightsquigarrow D$ is in $P(\mathcal{K})$.*

Hence, eventually, we have the completeness result.

Proposition 3 (Completeness). *Given a knowledge base $\mathcal{K}$, if a conditional $C \rightsquigarrow D$ is entailed by $\mathcal{K}$,* i.e. *$\mathcal{K} \models\!\!\approx C \rightsquigarrow D$, then $C \rightsquigarrow D$ is in $P(\mathcal{K})$.*

Corollary 1. *Given a knowledge base $\mathcal{K}$, $\mathcal{K} \models\!\!\approx C \rightsquigarrow D$ iff $C \rightsquigarrow D \in P(\mathcal{K})$.*

Extended Preferential Entailment. In the following we make one additional step by extending preferential entailment over Gödel logic with the aim to capture a missing property of classical preferential entailment, as the one illustrated below. Specifically, let us note that the following property (S) is derivable from the preferential properties in the classical propositional case:

$$\text{(S)}\ \frac{C \wedge D \rightsquigarrow E}{C \rightsquigarrow D \supset E}$$

Unfortunately, in the case of Gödel logic we can not derive it anymore from our rules.

Example 4 (Example 3 cont.). Consider Example 3. We have seen that we may infer

$$rf \wedge tg \rightsquigarrow s$$

In a classical preferential setting we may infer

$$rf \rightsquigarrow tg \supset s\ ,$$

while under preferential Gödel logic we can not. □

So, we next consider the non-monotonic conditional defined by the previous rules, *i.e.* (REF), (LLE), (RW), (CT), (CM), (OR) and (DN), with the addition of (S). Let us call P' both the set of such rules and the closure operation defined by such a set of rules. Our goal is now to semantically characterise P'.

Luckily, the semantic characterisation of P' is easily obtained as it is sufficient to constrain the previous bs-interpretations to the ones in which the belief sets correspond to singleton sets, *i.e.* single valuations.

Definition 6 (Preferential interpretations). *A* preferential interpretation *is a triple $M = \langle \mathcal{S}, \ell, \prec \rangle$, with $\mathcal{S}$ a set of states, $\ell : \mathcal{S} \to \mathcal{I}$ a function that associates to every state s a valuation $u \in \mathcal{I}$, and $\prec$ a preferential relation (asymmetric and transitive), that satisfies the property of smoothness.*

Note that this new class of interpretations is not properly a subclass of the interpretations based on the belief states, since here it is possible to have the same valuation present more than once in a model (we could have that two states $s, t \in \mathcal{S}$ are associated to the same valuation, *i.e.*, $\ell(s) = \ell(t)$), while in the belief-states proposal a subset of $\mathcal{I}$ can appear at most once in a model. Therefore, we have to redefine some previous notions in order to deal with the new kind of models.

To start with, again, $\hat{C}$ will be *extension* of C in M, *i.e.* the set of the states in M that are associated to a valuation verifying a proposition C to a positive degree: that is,

$$\hat{C} = \{s \in \mathcal{S} \mid \ell(s)(C) > 0\}\ .$$

Similarly, $\overline{C}$ is the set of the preferred states in the extension of C, that is

$$\overline{C} = \min_{\prec}(\hat{C}) = \{s \in \hat{C} \mid \nexists t \in \hat{C} \text{ such that } t \prec s\}\ .$$

We say that M *verifies* a proposition C, denoted $M \models\!\approx' C$, iff for each $s \in \mathcal{S}$, $\ell(s)(C) = 1$. Moreover, M *verifies* a conditional $C \rightsquigarrow D$, denoted $M \models\!\approx' C \rightsquigarrow D$, iff for every

$s \in \overline{C}$, $\ell(s)(C \supset \neg\neg D) = 1$. We say that M is a *preferential model* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ ($M \approx' \mathcal{K}$) iff $M \approx' E$ for every $E \in \mathcal{T}$ and $M \approx' E \rightsquigarrow F$ for every $E \rightsquigarrow F \in \mathcal{D}$. Eventually, we shall indicate with $\approx'$ the entailment relation defined using preferential models.

Definition 7 (Consequence relation $\approx'$). *A proposition C is* (preferentially) entailed *by $\mathcal{K}$, denoted $\mathcal{K} \approx' C$, iff for every preferential model M of $\mathcal{K}$, $M \approx' C$ holds. A conditional $C \rightsquigarrow D$ is* (preferentially) entailed *by $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, denoted $\mathcal{K} \approx' C \rightsquigarrow D$, iff for every preferential model M of $\mathcal{K}$, $M \approx' C \rightsquigarrow D$ holds.*

Like the previous section, we want again to prove that the closure operation P' is complete w.r.t. the consequence relation $\approx'$, that is

$$P'(\mathcal{K}) = \{C \rightsquigarrow D \mid \mathcal{K} \approx' C \rightsquigarrow D\} \ .$$

In this case, the completeness proof follows quite faithfully the representation proof for propositional classical logic in [17]. We have only to consider some contextual changes due to the different underlying monotonic consequence relation (the one defining propositional Gödel logic instead of the one associated to classical propositional logic) and the presence of the two extra-axioms (DN) and (S).

Indeed, we can show that we can obtain a completeness result. Being the proof very similar to the one in [17], we omit here the list of its main steps.

Proposition 4. *Given a finite set of conditionals $\mathcal{K}$, a conditional $C \rightsquigarrow D$ is in $P'(\mathcal{K})$ iff $\mathcal{K} \approx' C \rightsquigarrow D$.*

4 Rational Monotony

Another property that has been deeply investigated in non-monotonic logic is *Rational Monotony* (RM), namely

$$\text{(RM)} \ \frac{C \rightsquigarrow E \quad C \not\rightsquigarrow \neg D}{C \wedge D \rightsquigarrow E}$$

Rational Monotony is a form of constrained monotony that is stronger than (CM). Intuitively, it states that if typically the truth value of C is connected to the truth value of E, while a typical situation in $\hat{C}$ does not force $\neg D$ to be true, then in a typical situation in which $C \wedge D$ has a positive degree of truth also E is true to a positive degree.

Example 5 (Example 4 cont.). Consider Example 4. According to (RM) we may infer that “typically, a ripe and expensive fruit is sweet”, that is, from $rf \rightsquigarrow s$ and $rf \not\rightsquigarrow \neg e$ (e stands for expensive), we may infer via (RM) that

$$rf \wedge e \rightsquigarrow s \ .$$

This inference is not supported by preferential entailment. □

In order to semantically characterise the property (RM) we have to add a new constraint to the preferential order $\prec$ in the interpretation, that is, *modularity*.

Definition 8 (Modularity). *A partial order $\prec$ on a set $\mathcal{S}$ is* modular *if for every $x, y, z \in \mathcal{S}$, if $x \prec y$, then either $z \prec y$ or $x \prec z$.*

Informally, a modular order organises the elements of the set into layers, and all the elements of a lower layer are preferred to all the elements laying in higher layers. In our context, we will take under consideration the class of the preferential interpretations that have a modular preference order, that, following [18], we call *ranked* interpretations.

Definition 9 (Ranked Gödel interpretations). *A* ranked interpretation *is a triple $M = \langle \mathcal{S}, \ell, \prec \rangle$, with $\mathcal{S}$ a set of states, $\ell : \mathcal{S} \to \mathcal{I}$ a function that associates to every state s a valuation $u \in \mathcal{I}$, and $\prec$ a modular relation, that satisfies the property of smoothness.*

Now, it can be verified that the class of the ranked interpretations satisfy the property (RM).

Proposition 5 (Soundness). *The properties in P' and (RM) are verified by the class of ranked Gödel interpretations.*

However, we cannot define a form of entailment based on the ranked interpretations as we have done in the preferential case, as it may not give any inferential gain. Indeed, let us say that ranked interpretation M is a *ranked model* of a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ iff $M \approx' C$ for every $C \in \mathcal{T}$ and $M \approx' E \rightsquigarrow F$. Then

Definition 10 (Consequence relation $\approx''$). *A proposition C is* (rationally) entailed *by $\mathcal{K}$, denoted $\mathcal{K} \approx'' C$, iff for every ranked model M of $\mathcal{K}$, $M \approx' C$ holds. A conditional $C \rightsquigarrow D$ is* (rationally) entailed *by $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, denoted $\mathcal{K} \approx'' C \rightsquigarrow D$, iff for every ranked model M of $\mathcal{K}$, $M \approx' C \rightsquigarrow D$ holds.*

Then we can prove that such an entailment relation corresponds to the closure operation P'. That is,

Proposition 6. *$\mathcal{K} \approx'' C \rightsquigarrow D$ iff $C \rightsquigarrow D \in P'(\mathcal{K})$.*

Therefore, the entailment relation $\approx''$, does not provide any inferential gain over $\approx'$.

4.1 Rational Closure

Since it is not possible to define a form of non-monotonic reasoning that satisfies the rule (RM) and is based on a classical form of entailment, *i.e.*, defined considering all the ranked models of the knowledge base, Lehmann and Magidor [18] have indicated a form of non-monotonic logical closure of the knowledge base, called *Rational Closure* (RC), that satisfies a series of desiderata and is defined considering only some relevant ranked models of the knowledge base. We shall indicate by $R(\mathcal{K})$ the rational closure of the knowledge base $\mathcal{K}$.

Considering the results in the previous section, it is easy to see that the definition of Lehmann and Magidor's decision procedure is applicable also w.r.t. our preferential semantics and our conditional. From the semantical point of view, we shall refer to the semantic construction of Rational Closure by Giordano et al. [12] that we find more intuitive than the original formulation by Lehmann and Magidor.

The first step of the procedure is the definition of the notion of *exceptionality*.

Exceptionality. A proposition is exceptional if it is falsified in all the most typical situations that satisfy a knowledge base. That is, a proposition C is exceptional w.r.t. a knowledge base $\mathcal{K}$ iff it is falsified in all the preferential models of the knowledge base, *i.e.*, $\top \rightsquigarrow \neg C \in P'(\mathcal{K})$. The decision whether a proposition is exceptional can be reduced to a fuzzy entailment decision problem. In fact, we interpret every non-monotonic conditional $C \rightsquigarrow D$ as the satisfaction of $C \supset \neg\neg D$ in the most typical situations; we shall indicate with $\mathcal{D}^{\supset}$ the set of the material implications corresponding to the conditionals in the knowledge base. That is, given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$,

$$\mathcal{D}^{\supset} := \{C \supset \neg\neg D \mid C \rightsquigarrow D \in \mathcal{D}\} .$$

Such a set will be used to decide exceptionality as a classical decision problem.

Proposition 7. *Given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$,*

$$\top \rightsquigarrow C \in P'(\mathcal{K}) \textit{ iff } \mathcal{T} \cup \mathcal{D}^{\supset} \models \neg\neg C .$$

A conditional $C \rightsquigarrow D$ is exceptional if its antecedent C is exceptional. Hence, we can define a function $\mathcal{E}$ that, given $\langle \mathcal{T}, \mathcal{D} \rangle$, gives back the set of the exceptional conditionals in $\mathcal{D}$, that is,

$$\mathcal{E}(\langle \mathcal{T}, \mathcal{D} \rangle) := \{C \rightsquigarrow D \in \mathcal{D} \mid \mathcal{T} \cup \mathcal{D}^{\supset} \models \neg C\} .$$

The construction of the Rational Closure of a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$ is then based on the notion of exceptionality. That is, we can create a ranking of the conditionals in $\mathcal{D}$ using the function $\mathcal{E}$. To this end, we define a sequence of subsets of $\mathcal{D}$ in the following way:

$$\begin{aligned} E_0 &:= \mathcal{D} \\ E_{i+1} &:= \mathcal{E}(E_i) . \end{aligned}$$

Since the set $\mathcal{D}$ is finite, and every application of $\mathcal{E}$ on a set X gives back a subset of X, the procedure ends into an (empty or non-empty) fixed-point of the function $\mathcal{E}$, that we shall call E_∞.

Now, we can partition the set $\mathcal{D}$ in to a sequence $\langle \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{D}_\infty \rangle$, where $\mathcal{D}_i := E_i \setminus E_{i+1}$ $(0 \leq i \leq n)$ and $\mathcal{D}_\infty := E_\infty$. Each set $\mathcal{D}_i$ will contain the conditionals that have i as *ranking value*, starting from the conditionals in $\mathcal{D}_0$, describing what is verified only in the most normal situations, up to $\mathcal{D}_\infty$, describing what does not hold even in the most exceptional situations.

Note that, assuming that the cardinality of $\mathcal{D}$ is m, the identification of the partition $\langle \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{D}_\infty \rangle$ is definable doing $O(m^2)$ fuzzy entailment tests for propositional Gödel logic, and, for a given knowledge base, once such partition is done, it is done once for all.

Now we can define the ranking value of every formula in our language using the partition of $\mathcal{D}$ into $\mathcal{D}_0, \ldots, \mathcal{D}_n, \mathcal{D}_\infty$.

Definition 11 (Ranking value). *The ranking value of a proposition C is i, denoted $rank(C) = i$, iff $\mathcal{D}_i$ is the first element of the sequence $\langle \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ s.t.*

$$\langle \mathcal{T}, \mathcal{D}_i \rangle \not\approx' \top \rightsquigarrow \neg C .$$

If there is not such an element, $rank(C) = \infty$. The ranking value of a conditional $C \rightsquigarrow D$, denoted $rank(C \rightsquigarrow D)$, is the ranking value of C, i.e. $rank(C \rightsquigarrow D) = rank(C)$.

Note that, due to Proposition 7, the decision of the ranking value of a formula can be determined in $O(m)$ fuzzy entailment tests.

Following [18], a conditional $C \rightsquigarrow D$ is in the rational closure of the knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$ if the ranking value of $C \wedge D$ is lower than the ranking value of $C \wedge \neg D$, that is, the situation in which $C \wedge D$ has a positive degree of truth is less exceptional than the situation in which $C \wedge \neg D$ has a positive degree of truth. That is, we now can formulate the "somewhat typical" definition involving the constraint on ranks (see also the condition on possibility distributions in the introduction)

Definition 12 (Rational Closure). *$C \rightsquigarrow D \in R(\mathcal{K})$ iff either $rank(C \wedge D) < rank(C \wedge \neg D)$ or $rank(C) = \infty$.*

Example 6 (Example 5 cont.). Let's check whether we can infer that

$$rf \wedge e \rightsquigarrow s\ .$$

First of all we have to calculate the ranking value of the conditional $rf \rightsquigarrow s$. Since $\mathcal{T} \cup \{rf \supset \neg\neg s\} \not\models \neg rf$, $rank(rf \rightsquigarrow s) = rank(rf) = 0$ and $\mathcal{D}$ is partitioned into a single set $\mathcal{D}_0 = \mathcal{D}$. Now we have to check the ranking values of

$$rf \wedge e \wedge s \ \text{ and } \ rf \wedge e \wedge \neg s\ .$$

We have that $\mathcal{T} \cup \mathcal{D}_0^{\supset} \not\models \neg(rf \wedge e \wedge s)$ and, thus, $rank(rf \wedge e \wedge s) = 0$, while $\mathcal{T} \cup \mathcal{D}_0^{\supset} \models \neg(rf \wedge e \wedge \neg s)$, since $\mathcal{D}_0^{\supset} = \{rf \supset \neg\neg s\}$ and, thus, $rank(rf \wedge e \wedge \neg s) > 0$.

From these ranking values, we can conclude that

$$rf \wedge e \rightsquigarrow s \in R(\mathcal{K})\ .$$

□

Please observe that, since all computations are based on a polynomially bounded number of fuzzy entailment tests, the computational complexity of the decision procedure for Rational Closure is the same as the entailment problem for propositional Gödel logic and the procedure can be implemented once a decision procedure for fuzzy logic entailment is at hand.

Proposition 8. *Deciding whether $C \rightsquigarrow D \in R(\mathcal{K})$ is a coNP-complete problem.*

Semantic characterisation. We now give also a semantic characterisation of the above construction, still referring to the analogous constructions for classical propositional logic. A nice and intuitive characterisation of Rational Closure is given using the *minimal ranked models* introduced in [12]. We apply here a similar definition related to propositional Gödel logic.

The intuitive idea is the following: given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$, we consider all the ranked Gödel interpretations satisfying $\langle \mathcal{T}, \mathcal{D} \rangle$ that are *compatible* with $\langle \mathcal{T}, \mathcal{D} \rangle$, *i.e.* all the valuations v that verify $\mathcal{T} \cup \{\neg C \mid C \rightsquigarrow D \in \mathcal{D}_\infty\}$. Among all such models, we prefer those models in which all the valuations are considered 'as much typical as possible', that is, in which the valuations are ranked as low as possible.

First of all we need to define the *height* of a state $s \in \mathcal{S}$ in a ranked interpretation $M = \langle \mathcal{S}, \ell, \prec \rangle$.

Definition 13 **(Height k).** *Consider a ranked interpretation $M = \langle \mathcal{S}, \ell, \prec \rangle$, with $s \in \mathcal{S}$. The height $k_M(s)$ of s is the length of any chain $s_0 \prec \ldots \prec s$ from a s_0 s.t. for no $s' \in \mathcal{S}$ it holds that $s' \prec s_0$ [7]. The height of a formula C, $k_M(C)$, corresponds to the height of the states with the lowest height that do not falsify C, that is, $k_M(C) = k_M(s)$ s.t. $v(C) > 0$, where $\ell(s) = v$, and there is no state s' s.t. $s' \prec s$ and $v'(C) > 0$, where $\ell(s') = v'$.*

Note that it is easy to see that $M \approx\!\!\!\!\!| \; C \rightsquigarrow D$ iff $k_M(C \wedge D) < k_M(C \wedge \neg D)$ (it is immediate to check that in Gödel logic $v(C) > 0$ iff $v(\neg\neg C) > 0$, for any valuation v and any formula C, and hence $k_M(C \wedge D) = k_M(C \wedge \neg\neg D)$).

Definition 14 **(Minimal Ranked Models).** *Consider two ranked models of $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, $M = \langle \mathcal{S}, \ell, \prec \rangle$ and $M' = \langle \mathcal{S}, \ell, \prec' \rangle$. $\mathcal{S}$ and ℓ are such that for every valuation v compatible with $\mathcal{K}$ there is a state $s \in \mathcal{S}$ s.t. $\ell(s) = v$. We say that M is at least as preferred as M' ($M \leq_R M'$) iff for each $s \in \mathcal{S}$, $k_M(s) \leq k_{M'}(s)$. Let $\mathfrak{M}^R_{\mathcal{K}}$ be the set of the minimal ranked models of the knowledge base $\mathcal{K}$, that is, $\mathfrak{M}^R_{\mathcal{K}} = \{M \mid M \approx\!\!\!\!\!| \;' \mathcal{K}$ and $\nexists M'$ s.t. $M' \approx\!\!\!\!\!| \;' \mathcal{K}$ and $M' \leq_R M\}$.*

We define *minimal ranked entailment*, denoted $\approx\!\!\!\!\!| \;_R$, as the entailment relation defined by the minimal ranked models.

Definition 15 **(Minimal Ranked Entailment).** *A conditional $C \rightsquigarrow D$ is a minimal ranked consequence of a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, denoted $\mathcal{K} \approx\!\!\!\!\!| \;_R C \rightsquigarrow D$, iff every ranked model in $\mathfrak{M}^R_{\mathcal{K}}$ verifies $C \rightsquigarrow D$.*

We can prove that this notion of entailment characterises the closure operation R.

Proposition 9. *Given a knowledge base $\mathcal{K}$, $C \rightsquigarrow D \in R(\mathcal{K})$ iff $\mathcal{K} \approx\!\!\!\!\!| \;_R C \rightsquigarrow D$.*

The proof of Proposition 9 follows the proof of the analogous result in [12, 13], reformulated in order to take into account that we are dealing with Gödel logic, that the conditional $C \rightsquigarrow D$ is interpreted w.r.t. the formula $C \supset \neg\neg D$ and that there are also the rules (DN) and (S) to take into account.

5 Conclusions

The notion of rational closure is acknowledged as a landmark for defeasible reasoning, while mathematical fuzzy logic is the reference framework to deal with fuzziness. In this work we have made a first attempt to connect the two, by characterising rational closure in the context of Propositional Gödel Logic, axiomatically, semantically, algorithmically and from a computational complexity point of view.

We plan to continue our investigation along several directions. Specifically, to extend our approach towards other fuzzy logics, such as Łukasiewicz and Product logics, to extend it to notable fragments of First-Order Logic, such as fuzzy Description Logics [19, 22] along the line [5], and to investigate about possible connections to a possibilistic logic based approach in line with [2–4, 6–8, 10] including as well different interpretations of fuzzy implications as discussed in [9].

[7] Note that for ranked interpretations, $k_M(s)$ is uniquely determined. See also [13].

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Appendix: Proofs

A Proofs for Section 3

Lemma 1 *The conditional* $\rightsquigarrow$ *satisfies* supraclassicality *(SUPRA):*

$$\frac{C \supset D}{C \rightsquigarrow D}$$

Proof. From (REF) and (RW).

Lemma 2 *If a conditional* $\rightsquigarrow$ *satisfies the properties defining the closure operation* P*, then it satisfies also the following properties:*

$$\text{(EQUIV)}\ \frac{C \rightsquigarrow D \quad D \rightsquigarrow C \quad C \rightsquigarrow E}{D \rightsquigarrow E} \qquad \text{(AND)}\ \frac{C \rightsquigarrow D \quad C \rightsquigarrow E}{C \rightsquigarrow D \wedge E}$$

$$\text{(MPC)}\ \frac{C \rightsquigarrow D \supset E \quad C \rightsquigarrow D}{C \rightsquigarrow E} \qquad (1)\ \frac{C \vee D \rightsquigarrow C \quad C \rightsquigarrow E}{C \vee D \rightsquigarrow E}$$

$$(2)\ \frac{C \rightsquigarrow E \quad D \rightsquigarrow F}{C \vee D \rightsquigarrow E \vee F} \qquad (3)\ \frac{C \rightsquigarrow D}{C \rightsquigarrow \neg\neg D}$$

$$(4)\ \frac{C \vee D \rightsquigarrow C \quad D \vee E \rightsquigarrow D}{C \vee E \rightsquigarrow C}$$

Proof. (EQUIV): apply (CM) to obtain $C \wedge D \rightsquigarrow E$ and then (CT).

(AND): from (CM) we have $C \wedge D \rightsquigarrow E$, and since $C \wedge D \wedge E \supset D \wedge E$ is verified, using (SUPRA) we obtain $C \wedge D \wedge E \rightsquigarrow D \wedge E$. We apply (CT) to the latter and $C \wedge D \rightsquigarrow E$ and we obtain $C \wedge D \rightsquigarrow D \wedge E$, and, again by (CT) using $C \rightsquigarrow D$, $C \rightsquigarrow D \wedge E$.

(MPC): we can easily check that in propositional Gödel logic we have $\models (C \wedge (C \supset D)) \supset D$: for any valuation u, if $u(C) > u(D)$, then $u(C \supset D) = u(D)$, and consequently $u(C \wedge (C \supset D)) = u(D)$. Eventually, $u(C \wedge (C \supset D)) \supset D) = u(D \supset D) = 1$. If $u(C) \leq u(D)$, then $u(C \supset D) = 1$, $u(C \wedge (C \supset D)) = u(C)$ and, eventually, $u(C \wedge (C \supset D)) \supset D) = u(D \supset D) = 1$. Given this, we obtain (MPC) from (AND) and (RW).

(1): From (SUPRA) we have $C \rightsquigarrow C \vee D$, and we can apply (EQUIV).

(2): Apply (RW) to each premise and then (OR).

(3): In Gödel logic $C \supset \neg\neg C$ is valid, hence we prove (3) using (RW)

(4): Applying (2) to the premises we obtain $C \vee D \vee E \rightsquigarrow C \vee D$, that with the first premise and (1) gives $C \vee D \vee E \rightsquigarrow C$. Applying (2) to the first premise and $E \rightsquigarrow E$ we have $C \vee D \vee E \rightsquigarrow C \vee E$. Apply (CM) to the latter and $C \vee D \vee E \rightsquigarrow C$ and we have $C \vee E \rightsquigarrow C$.

Proposition 2 (Soundness) *Given a knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$*, if a conditional* $C \rightsquigarrow D$ *is in* $P(\mathcal{K})$ *then* $\mathcal{K} \models\!\!\approx C \rightsquigarrow D$.

Proof. It is immediate to see that a preferential model verifies the strict part $\mathcal{T}$ of the knowledge base iff its belief sets contain only valuation that verify the propositions in $\mathcal{T}$. Hence, the set of the preferential interpretations of $\mathcal{K}$ satisfy all the theorems of the consequence relation $\models_{\mathcal{T}}$.

Then, it is sufficient to prove that the structural properties are satisfied by $\models\!\approx$.

For (REF), consider that $C \supset \neg\neg C$ is a valid formula in Gödel logic.

For (LLE) the proof is trivial.

For (RW), $M \models\!\approx C \rightsquigarrow D$ means that for every valuation u in the belief states in $\overline{C}$, $u(C \supset \neg\neg D) = 1$; since $u(C) > 0$, the latter implies that $u(D) > 0$, that in turn, since $\models_{\mathcal{T}} D \supset E$, implies that $u(E) > 0$, *i.e.*, $u(\neg\neg E) = 1$.

(CM) and (CT): If a preferential interpretation M satisfies $C \rightsquigarrow D$, then $\overline{C} \subseteq \widehat{\neg\neg D}$, *i.e.*, $\overline{C} \subseteq \hat{D}$. Consequently $\overline{C \wedge D} = \overline{C}$. This proves the satisfaction of both (CM) and (CT).

About (OR), consider that $(\widehat{C \vee D}) = \hat{C} \cup \hat{D}$, hence we can easily see that $\overline{C \vee D} \subseteq \overline{C} \cup \overline{D}$, proving the satisfaction of $C \vee D \rightsquigarrow E$.

Lemma 3 *For every proposition D, $u(D) > 0$ for all the valuations that are normal for C w.r.t. $\mathcal{K}$ iff $C \rightsquigarrow D \in P(\mathcal{K})$.*

Proof. $\Leftarrow$) it follows immediately from the definition of normal valuation, since for every valuation u, $u(\neg\neg D) = 1$ iff $u(D) > 0$.

$\Rightarrow$) we have to prove that if $C \rightsquigarrow D \notin P(\mathcal{K})$, there is a normal valuation u for C s.t. $u(D) = 0$. Let $\mathcal{C}^{\rightsquigarrow}$ be the set of the double negations of all the typical consequences of C, that is, $\mathcal{C}^{\rightsquigarrow} = \{\neg\neg E \mid C \rightsquigarrow E \in P(\mathcal{K})\}$. If there is no valuation u that is normal for C and s.t. $u(\neg\neg D) = 0$, it means that there is a finite subset $\mathcal{C}'$ of $\mathcal{C}^{\rightsquigarrow}$ s.t. $\models_{\mathcal{T}} (C \wedge \bigwedge \mathcal{C}') \supset \neg\neg D$, that would imply by (SUPRA) (Lemma 1) that $(C \wedge \bigwedge \mathcal{C}') \rightsquigarrow \neg\neg D \in P(\mathcal{K})$. From this and various applications of the rule (3) of Lemma 2 and of (CT), we obtain $C \rightsquigarrow \neg\neg D \in P(\mathcal{K})$. Eventually, applying (DN) we have $C \rightsquigarrow D \in P(\mathcal{K})$, against the hypothesis.

Lemma 4 *$C \sim D \in P(\mathcal{K})$ iff for every proposition E, $C \rightsquigarrow E \in P(\mathcal{K})$ iff $D \rightsquigarrow E \in P(\mathcal{K})$.*

Proof. $\Leftarrow$) It follows immediately from Reflexivity.

$\Rightarrow$) It follows from (EQUIV) (Lemma 2).

Lemma 5 *If $\rightsquigarrow$ is a preferential conditional, then $<$ is asymmetric and transitive.*

Proof. Asymmetry comes automatically from the definition of $<$, while transitivity follows from the transitivity of $\leq$, that follows from (4).

Lemma 7 *Given $M^{\mathcal{K}}$, for every proposition C, $\overline{C} = \{[C^{\sim}]\}$.*

Proof. Firstly, we prove that $[C^{\sim}] \in \overline{C}$: if $[D^{\sim}] \prec^{\mathcal{K}} [C^{\sim}]$, then $D \rightsquigarrow C \notin P(\mathcal{K})$, otherwise by (OR) we would have $C \vee D \rightsquigarrow C$, against $[D^{\sim}] \prec^{\mathcal{K}} [C^{\sim}]$. From $D \rightsquigarrow C \notin P(\mathcal{K})$ and Lemma 3 we have that $[D^{\sim}] \notin \hat{C}$, and consequently $[C^{\sim}] \in \overline{C}$.

Now we have to prove that $[C^{\sim}]$ is the only member of $\overline{C}$. Assume there is a $[D^{\sim}] \in \overline{C}$ s.t. $[D^{\sim}] \neq [C^{\sim}]$. It must be $[D^{\sim}] \not\prec^{\mathcal{K}} [C^{\sim}]$ and $[C^{\sim}] \not\prec^{\mathcal{K}} [D^{\sim}]$, hence either $C \leq D$ and $D \leq C$, or $C \not\leq D$ and $D \not\leq C$.

In the former case we have $C \vee D \rightsquigarrow C$ and $C \vee D \rightsquigarrow D$, that by (CM) and (LLE) would imply $C \rightsquigarrow D$ and $D \rightsquigarrow C$, and hence $[D^{\sim}] = [C^{\sim}]$ against the hypothesis.

If $C \nleq D$ and $D \nleq C$, then we can conclude that $D \rightsquigarrow C \notin P(\mathcal{K})$, otherwise from $C \rightsquigarrow C$, (REF) and (OR) we would have $C \vee D \rightsquigarrow C$ and $C \leq D$. Hence, by Lemma 3, $[D^{\sim}] \notin \hat{C}$.

Lemma 8 *$M^{\mathcal{K}}$ is a preferential interpretation.*

Proof. We have to prove that $\prec^{\mathcal{K}}$ is asymmetric and transitive, and satisfies the smoothness condition. Asymmetry and transitivity follow immediately from Lemma 5, while Lemma 7 guarantees smoothness, since for every C, $\overline{C} = [C^{\sim}]$.

Lemma 9 *Given a knowledge base $\mathcal{K}$, for every conditional $C \rightsquigarrow D$, $M^{\mathcal{K}} \models\!\!\approx C \rightsquigarrow D$ implies $C \rightsquigarrow D$ is in $P(\mathcal{K})$.*

Proof. It is immediate from Lemma 3 and Lemma 7.

Proposition 3 (Completeness). *Given a knowledge base $\mathcal{K}$, if a conditional $C \rightsquigarrow D$ is entailed by $\mathcal{K}$,* i.e. *$\mathcal{K} \models\!\!\approx C \rightsquigarrow D$, then $C \rightsquigarrow D$ is in $P(\mathcal{K})$.*

Proof. It is a direct consequence of Proposition 2, Lemma 9 and Lemma 8.

Now, moving to the extended preferential entailment, we want to prove the following theorem.
Proposition 4 (Completeness). *Given a finite set of conditionals $\mathcal{K}$, a conditional $C \rightsquigarrow D$ is in $P'(\mathcal{K})$ iff $\mathcal{K} \models\!\!\approx' C \rightsquigarrow D$.*
The proof follows the original proof for propositional logic in [17]. Here below we present the main lemmas needed in order to obtain the completeness result. First, some structural properties of $\rightsquigarrow$ that are valid in P', and that will be necessary in the proofs.

Lemma 10. *If a conditional $\rightsquigarrow$ satisfies the properties in P', then it satisfies also the following properties:*

$$(5)\ \frac{C \vee D \rightsquigarrow C \quad D \vee E \rightsquigarrow D}{C \rightsquigarrow E \supset D} \qquad (6)\ \frac{C \vee E \rightsquigarrow E \quad C \rightsquigarrow D}{E \rightsquigarrow C \supset D}$$

Proof. (5): From $D \vee E \rightsquigarrow D$ and (LLE) we have $(D \vee E) \wedge (C \vee D \vee E) \rightsquigarrow D$. By (S), $(C \vee D \vee E) \rightsquigarrow (D \vee E) \supset D$, and, by (RW), $(C \vee D \vee E) \rightsquigarrow E \supset D$. From the two premises and (2) we obtain $C \vee D \vee E \rightsquigarrow C \vee D$. From the latter and $(C \vee D \vee E) \rightsquigarrow E \supset D$ we obtain $(C \vee D) \rightsquigarrow E \supset D$ by (CM), and applying (CM) again with the first hypothesis and (LLE) we conclude $C \rightsquigarrow E \supset D$.

(6): from $C \rightsquigarrow E$ and (LLE), $(C \vee E) \wedge C \rightsquigarrow E$, that, using (S), gives $(C \vee E) \rightsquigarrow C \supset E$. Using $(C \vee E) \rightsquigarrow E$, apply (CM) and (LLE) and obtain $E \rightsquigarrow C \supset D$.

Now the soundness result.

Proposition 10 (Soundness). *Given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, if a conditional $C \rightsquigarrow D$ is in $P'(\mathcal{K})$ then $\mathcal{K} \models\!\!\approx' C \rightsquigarrow D$.*

Proof. Even if the present interpretations are not exactly special cases of the models based on the Belief States (there can be in the same model more than one state associated to the same valuation), it is easy to prove the validity of the properties (REF)-(OR) in these interpretations following the proofs for Proposition 2.

For (S) we take under consideration every valuation u in $\overline{C}$ and we have to check two cases:

- $u \in \overline{C}$ and $u \notin \hat{D}$. Hence $u(D) = 0$ and consequently $u(D \supset \neg\neg E) = 1$.
- $u \in \overline{C}$ and $u \in \hat{D}$. Then $u \in \overline{C \wedge D}$ and $u(\neg\neg E) = 1$. Since $u(D \supset \neg\neg E)$ is equal either to 1 or to $u(\neg\neg E)$, then $u(D \supset \neg\neg E) = 1$.

Now we proceed to the proof of the completeness of P' w.r.t. $\models\!\!\approx'$. First, analogously to the case of the belief sets, let's define the notion of *normal valuation* w.r.t. a proposition C, that is, a valuation that makes true all the conditionals in $P(\mathcal{K})$ that have C as antecedent.

Definition 16 (Normal valuation). *A valuation u is* normal *for a proposition C w.r.t a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ iff $u(C) > 0$, for every proposition $E \in \mathcal{T}$ $u(E) = 1$, and for every proposition D s.t. $C \rightsquigarrow D \in P(\mathcal{K})$, $u(C \supset \neg\neg D) = 1$ (*i.e., *$u(\neg\neg D) = 1$).*

Now, we prove a main lemma, that states that taking under consideration all the normal valuation for a proposition C we are able to characterise the closure P w.r.t. C.

Lemma 11. *For every proposition D, $u(D) > 0$ for all the valuations that are normal for C w.r.t. $\mathcal{K}$ iff $C \rightsquigarrow D \in P'(\mathcal{K})$.*

Proof. It is analogous to the proof of Lemma 3.

As for the closure P, if $C \rightsquigarrow D$ and $D \rightsquigarrow C$ are both in $P'(\mathcal{K})$, then we say that C and D are *preferentially equivalent*, and we indicate it by $C \sim D \in P'(\mathcal{K})$.

Lemma 12. *$C \sim D \in P'(\mathcal{K})$ iff for every proposition E, $C \rightsquigarrow E \in P'(\mathcal{K})$ iff $D \rightsquigarrow E \in P'(\mathcal{K})$.*

Proof. The same proof as Lemma 4.

Now we define an ordering of the formulas w.r.t. the conditionals in the preferential closure $P(\mathcal{K})$.

Definition 17. *C is* not less ordinary *than D in $\mathcal{K}$, denoted $C \leq D$, iff $C \vee D \rightsquigarrow C \in P'(\mathcal{K})$.*

Lemma 13. *If $\rightsquigarrow$ is a preferential conditional, then $\leq$ is reflexive and transitive.*

Proof. Reflexivity is determined by (REF) ($C \rightsquigarrow C$) and (LLE) ($C \vee C \rightsquigarrow C$), while transitivity follows from (4).

Now we have to prove, given $\leq$, some properties of the normal valuations.

Lemma 14. *If $C \leq D$ and u is a normal valuation for C w.r.t. $\mathcal{K}$ s.t. $u(D) > 0$, then u is a normal valuation also for D w.r.t. $\mathcal{K}$.*

Proof. Suppose $D \rightsquigarrow E$. From $C \vee D \rightsquigarrow C$ and (6) we have that $C \rightsquigarrow D \supset E$, that is, $u(\neg\neg(D \supset E)) = 1$, *i.e.*, $u(D \supset E) \geq 0$. Since $u(D) > 0$, $u(E) > 0$ too, that is, $u(\neg\neg E) = 1$.

Lemma 15. *If $C \leq D \leq E$ and u is a normal valuation w.r.t. C s.t. $u(E) > 0$, then u is a normal valuation also for D.*

Proof. From (5) we have that $C \rightsquigarrow E \supset D$. This implies that $u(D) > 0$, and, by lemma 14, u is normal also for D.

Now we can define a model that is appropriate for proving completeness. Consider $M^{\mathcal{K}} = \langle \mathcal{S}^{\mathcal{K}}, \ell^{\mathcal{K}}, \prec^{\mathcal{K}} \rangle$, where:

- $\mathcal{S}^{\mathcal{K}} := \{\langle u, C \rangle \mid u \text{ is a normal valuation for } C\}$;
- $\ell^{\mathcal{K}}(\langle u, C \rangle) = u$;
- $\langle u, C \rangle \prec^{\mathcal{K}} \langle v, D \rangle$ iff $C \leq D$ and $u(D) = 0$.

We need to prove that $M^{\mathcal{K}}$ is a preferential model and some of its properties.

Lemma 16. $\prec^{\mathcal{K}}$ *is a partial order,* i.e.*, it is irreflexive and transitive.*

Proof. As for the propositional case ([17], Lemma 28).

Lemma 17. *In $M^{\mathcal{K}}$ $\langle u, C \rangle$ is in $\overline{D}$ iff $u(D) > 0$ and $C \leq D$.*

Proof. $\Leftarrow$) Assume $\langle u, C \rangle \in \hat{D}$ and $C \vee D \rightsquigarrow C$. Suppose there is a state $\langle v, E \rangle$ s.t. $\langle v, E \rangle \prec^{\mathcal{K}} \langle u, C \rangle$ and $v(D) > 0$. Then it would be $E \leq C \leq D$, and, by Lemma 15, $v(C) > 0$, against the definition of $\prec^{\mathcal{K}}$.

$\Rightarrow$) Assume $\langle u, C \rangle \in \overline{D}$. Hence, $u(D) > 0$. Suppose v is a normal valuation for $C \vee D$ s.t. $v(C) = 0$. Since $C \vee D \leq C$, we have $\langle v, C \vee D \rangle \prec^{\mathcal{K}} \langle u, C \rangle$. But, since $v(C \vee D) > 0$ and $v(C) = 0$, it must be $v(D) > 0$, that is against the minimality of $\langle u, C \rangle$ in $\hat{D}$. Then for every normal valuation u for $C \vee D$, $u(C) > 0$ and, by Lemma 3, $C \vee D \rightsquigarrow C$ (*i.e.*, $C \leq D$).

At this point, we need to prove that $\prec^{\mathcal{K}}$ satisfies the smoothness condition.

Lemma 18. $\prec^{\mathcal{K}}$ *satisfies the smoothness condition.*

Proof. Let $\langle u, C \rangle \in \hat{D}$. If $C \leq D$, by Lemma 17 $\langle u, C \rangle \in \overline{D}$. Otherwise, if $C \not\leq D$ (*i.e.*, $C \vee D \rightsquigarrow C \notin P'(\mathcal{K})$) then by Lemma 3 there is a normal valuation v for $C \vee D$ s.t. $v(C) = 0$. But $C \vee D \leq C$, and therefore $\langle v, C \vee D \rangle \prec^{\mathcal{K}} \langle u, C \rangle$. Since $v(C \vee D) > 0$ and $v(C) = 0$, $v(D) > 0$. Since $C \vee D \leq D$, by Lemma 17 $\langle v, C \vee D \rangle$ is in $\overline{D}$.

Eventually, we can prove completeness.

Lemma 19. *If $\mathcal{K} \approx' C \rightsquigarrow D$, then $C \rightsquigarrow D \in P'(\mathcal{K})$.*

Proof. From Lemma 17 and Lemma 14 we have that in the preferential model $M^{\mathcal{K}}$ a state $\langle u, C \rangle$ is in $\overline{D}$ iff it is a normal valuation for D. From this and Lemma 11 we prove the statement.

Proposition 4 is a direct consequence of Proposition 10 and Lemma 19.

B Proofs for Section 4

Proposition 5 (Soundness). *The properties in P' and (RM) are satisfied by the class of ranked Gödel interpretations.*

Proof. For the properties in P' the proof is given, since it is a subclass of the preferential interpretations used in the previous section. For (RM), $C \not\rightsquigarrow \neg D$ implies that there is some valuation u in $\overline{C}$ s.t. $u(D) > 0$, *i.e.*, $\overline{C} \cap \hat{D} \neq \emptyset$. Hence, $\overline{C \wedge D} = \overline{C} \cap \hat{D}$, and $C \wedge D \rightsquigarrow E$.

Proposition 6. $\mathcal{K} \approx''C \rightsquigarrow D$ *iff* $C \rightsquigarrow D \in P'(\mathcal{K})$.

Proof. The proof is articulated, but it fully corresponds to the proofs of Lemma 4.1 and Theorem 4.2 in [18].

Proposition 7. *Given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, $\top \rightsquigarrow C \in P'(\mathcal{K})$ iff $\mathcal{T} \cup \mathcal{D}^{\supset} \models \neg\neg C$.*

Proof. $\Leftarrow$) Assume $\mathcal{T} \cup \mathcal{D}^{\supset} \models \neg\neg C$ and consider a preferential model of $\mathcal{K}$. Let $u \in \overline{\top}$. Clearly, $\overline{\top}$ is the set of the minimal valuations of the entire model. We want to prove that u verifies all the propositions in $\mathcal{T} \cup \mathcal{K}^{\supset}$. W.r.t. the propositions in $\mathcal{T}$, they must be satisfied in u, since we are dealing with a model of $\mathcal{K}$. Let $C \supset \neg\neg D \in \mathcal{K}^{\supset}$. If $u \in \overline{C}$, then $u \models C \supset \neg\neg D$ since $C \rightsquigarrow D \in \mathcal{K}$. If $u \notin \overline{C}$, then $u \notin \hat{C}$, since there are no valuations preferred to u, and automatically $u \models C \supset \neg\neg D$.

Hence, if $\mathcal{T} \cup \mathcal{D}^{\supset} \models \neg\neg C$, for every $u \in \overline{\top}$ we have that $u \models \top \supset \neg\neg C$. Hence every preferential model of $\mathcal{K}$ satisfies $\top \rightsquigarrow C$, and, by completeness, we have that $\top \rightsquigarrow C \in P'(\mathcal{K})$.

$\Rightarrow$) Assume that $\top \rightsquigarrow C \in P'(\mathcal{K})$ and that there is a valuation u s.t. $u(\neg\neg C) = 0$ (*i.e.*, $u(C) = 0$). Create a one-valuation interpretation with u, and you obtain a preferential interpretation that does not satisfy $\top \rightsquigarrow C$, against the hypothesis.

For Section 4.1, we want to prove the following proposition.

Proposition 9 *Given a knowledge base $\mathcal{K}$, $C \rightsquigarrow D \in R(\mathcal{K})$ iff $\mathcal{K} \approx_R C \rightsquigarrow D$.*

As specified in the main body of the paper, in order to prove the proposition it is sufficient to follow the proofs for the classical propositional logic in [13], reformulating them in order to deal with the underlying propositional Gödel logic and the present characterisation of the conditional.

We need to prove some lemmas in order to prove the main proposition.

Lemma 20. *Let $M = \langle \mathcal{S}, \ell, \prec \rangle$ be a ranked model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$. Let $M_0 = M$ and $M_i = \langle \mathcal{S}_i, \ell_i, \prec_i \rangle$ be the model obtained from M eliminating all the valuations v s.t. $k_M(v) < i$. For any formula C, if $rank(C) \geq i$ then (1) $k_M(C) \geq i$, and (2) if $E_i \approx' C \rightsquigarrow D$, then $M_i \approx C \rightsquigarrow D$.*

Proof. The proof is the same as in the classical case ([13], Proposition 5). it is sufficient to check the validity of Lemma 21 in the present setting.

Lemma 21. *Let i be an ordinal, $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ a conditional knowledge base and $C \rightsquigarrow D$ a conditional s.t. $rank(C \rightsquigarrow D) \geq i$. Then $C \rightsquigarrow D \in P'(\mathcal{K})$ iff $C \rightsquigarrow D \in P'(\langle \mathcal{T}, E_i \rangle)$.*

Proof. The proof is the same as the correspondent Lemma 2.21 in [18], based on the by induction on the length of the proof. We have to check just that it is valid also for (DN) and (S).

For (DN), assume $rank(C \rightsquigarrow \neg\neg D) \geq i$, *i.e.*, $rank(C) \geq i$. By inductive hypothesis we have that $C \rightsquigarrow \neg\neg D \in P'(\mathcal{K})$ iff $C \rightsquigarrow \neg\neg D \in P'(\langle \mathcal{T}, E_i \rangle)$. Now apply (DN); the resulting conditional $C \rightsquigarrow D$ has the same rank value as $C \rightsquigarrow \neg\neg D$, having the same antecedent, and the result is immediate.

For (S), assume $rank(C \wedge D \rightsquigarrow E) \geq i$, *i.e.*, $rank(C \wedge D) \geq i$. By inductive hypothesis we have that $C \wedge D \rightsquigarrow E \in P'(\mathcal{K})$ iff $C \wedge D \rightsquigarrow E \in P'(\langle \mathcal{T}, E_i \rangle)$. Now apply (S); we obtain the conditional $C \rightsquigarrow D \supset E$, where $rank(C) = j \leq i$ and we have that $C \rightsquigarrow D \supset E \in P'(\mathcal{K})$ iff $C \rightsquigarrow D \supset E \in P'(\langle \mathcal{T}, E_i \rangle)$. If $C \rightsquigarrow D \supset E \in P'(\langle \mathcal{T}, E_j \rangle)$ then $C \rightsquigarrow D \supset E \in P'(\mathcal{K})$ is immediate, and, since $E_j \supseteq E_i$, if $C \rightsquigarrow D \supset E \in P'(\mathcal{K})$ then $C \rightsquigarrow D \supset E \in P'(\langle \mathcal{T}, E_j \rangle)$.

Lemma 22. *Let $M = \langle \mathcal{S}, \ell, \prec \rangle$ be a minimal ranked model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ s.t. for every Gödel valuation v compatible with $\langle \mathcal{T}, \mathcal{D} \rangle$ there is a state $s \in \mathcal{S}$ s.t. $\ell(s) = v$. For all $s \in \mathcal{S}$ it holds that: if $v(C \supset \neg\neg D) = 1$ ($\ell(s) = v$) for all $C \rightsquigarrow D \in E_i$, then $k_M(s) \leq i$.*

Proof. It follows the proof of the classical counterpart ([13], Proposition 6). It is sufficient to reformulate the set S_x ([13], p. 4), where $x \in \mathcal{S}$, as $S_x = \{C \rightsquigarrow D \in \mathcal{D} \mid v(C \wedge \neg D) > 0, \text{ where } \ell(x) = v\}$.

Now we can prove the main lemma.

Lemma 23. *Let $M = \langle \mathcal{S}, \ell, \prec \rangle$ be a minimal ranked model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ s.t. for every Gödel valuation v compatible with $\langle \mathcal{T}, \mathcal{D} \rangle$ there is a state $s \in \mathcal{S}$ s.t. $\ell(s) = v$. For every formula C, $rank(C) = i$ iff $k_M(C) = i$.*

Proof. It follows the proof of the classical counterpart ([13], Proposition 7). We make use of Lemmas 20 and 22.

Given Lemma 23, the proof of Proposition 9 is immediate.